MA8402

PROBABILITY AND QUEUEING THEORY

UNIT 1 : PROBABILITY AND RANDOM VARIABLES

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PROBABILITY AND RANDOM VARIABLES

Definitions:

Deterministic experiments:-

There are experiments which always produce the same result.

Random experiments:-

The experiments which do not produce the same result.

Trial & event:-

The performance of a random experiment is called a trial & the outcome is called an event.

E.g: Throwing of a coin is a trial & getting H or T is an event.

Sample space:-

The totality of the possible outcomes of a random experiment is called the sample space of the experiment.

Equally likely events:-

The possibilities or events are said to be equally likely when we have no reason to expect any one rather than the other.

E.g: In tossing an unbiased coin, the head or tail are equally likely.

Mutually exclusive events:-

If A & B are mutually exclusive, then it is not possible for both events to occur on the same trial.

Exhaustive events:-

Events are said to be exhaustive when they include all possibilities.

Favourable events:-

The trails which entail the happening of an event are said to be favourable to the event.

Probability:- Chance of happening.

$$P(A) = \frac{\text{Favourable number of cases}}{\text{Exhaustive number of cases}}.$$

Permutation :-

Selection & arrangement of factors.

$$np_r = \frac{n!}{(n-r)!}$$

Permutations with repetitions :-

Let $p(n:n_1, n_2, \dots, n_r)$ denotes the number of permutations of n objects of which n_1 are alike, n_2 are alike, ..., n_r are alike, $p(n:n_1, n_2, \dots, n_r) = \frac{n!}{n_1!\, n_2! \dots n_r!}$.

Eg: The number of permutations of the word 'RADAR' is $\frac{5!}{2!\,2!} = 30$.

Combination:-

Combinations means selection of factors.

$$nc_r = \frac{np_r}{r!} = \frac{n!}{(n-r)!\,r!}.$$

Note:

$nc_n = nc_0 = 1$

$nc_r = nc_{n-r}$

Axioms of Probability:

If S is the sample space & E is any event in a random experiment,

Axiom 1: $0 \leq P(E) \leq 1$.

Axiom 2: $P(S) = 1$

Axiom 3: For any sequence of mutually exclusive events $E_1, E_2, \dots$,

$$P\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} P(E_i).$$

Problems:

① If 3 balls are randomly drawn from a bowl containing 6 white & 5 black balls, what is the probability that one of the drawn ball is white & the other two black?

Sol: P[One of the drawn ball is white & the other two are black] $= P(A) = \frac{n(A)}{n(S)}$

$$= \frac{6C_1 \times 5C_2}{11C_3} = \frac{60}{165} = \frac{4}{11}$$

② A lot of integrated circuit chips consists of 10 good, 4 with minor defects & 2 with major defects. Two chips are randomly chosen from the lot. What is the probability that atleast one chip is good?

Sol: P[atleast one chip is good] $= P(A) = \frac{n(A)}{n(S)}$

$$= \frac{(10C_1 \times 6C_1) + 10C_2}{16C_2} = \frac{60+45}{120} = \frac{105}{120} = \frac{7}{8}$$

③ 4 persons are chosen at random from a group containing 3 men, 2 women & 4 children. Show that the chance that exactly 2 of them will be children is $\frac{10}{21}$.

Sol: P[exactly 2 of them will be children] $= P(A) = \frac{n(A)}{n(S)}$

$$= \frac{4C_2 \times 5C_2}{9C_4} = \frac{60}{126} = \frac{10}{21}$$

④ From a group of 5 first year, 4 second year & 4 third year students, 3 students are selected at random. Find the probability that they are first year or third year students.

Sol:

P[They are first year or third year students] = $P(A) = \frac{n(A)}{n(S)}$

$$= \frac{5C_3 + 4C_3}{13C_3} = \frac{10+4}{286} = \frac{14}{286} = \frac{7}{143}$$

⑤ A coin is biased so that a head is twice as likely to occur as a tail. If the coin is tossed 3 times, what is the probability of getting 2 tails & 1 head.

Sol: Given $P(H) = \frac{2}{3}$ & $P(T) = \frac{1}{3}$.

$S = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$

$P(HTT) = \frac{2}{3} \times \frac{1}{3} \times \frac{1}{3} = \frac{2}{27}$; $P(THT) = \frac{2}{27}$; $P(TTH) = \frac{2}{27}$

$\therefore$ P[getting 2 tails & 1 head] $= \frac{2}{27} + \frac{2}{27} + \frac{2}{27} = \frac{6}{27}$

⑥ One card is drawn from a deck of 52 cards. What is the probability of the card being either red or a king?

Sol: WKT $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

P[the card being either red or a king] $= \frac{26C_1 + 4C_1 - 2C_1}{52C_1}$

$$= \frac{26+4-2}{52} = \frac{28}{52} = \frac{7}{13}$$

⑦ If A & B are events with $P(A) = \frac{3}{8}$, $P(B) = \frac{1}{2}$ & $P(A \cap B) = \frac{1}{4}$, find $P(A^c \cap B^c)$.

Sol:

$P(A^c \cap B^c) = P(\overline{A \cup B}) = 1 - P(A \cup B)$

$= 1 - [P(A) + P(B) - P(A \cap B)]$

$= 1 - \left[\frac{3}{8} + \frac{1}{2} - \frac{1}{4}\right] = 1 - \frac{5}{8} = \frac{3}{8}$

⑧ Let events A & B be independent with $P(A)=0.5$ & $P(B)=0.8$. Find the Probability that neither of the events A nor B occurs.

Sol:

$$P(\bar{A}\cap\bar{B}) = P(\overline{A\cup B}) = 1-P(A\cup B)$$

$$= 1-[P(A)+P(B)-P(A\cap B)]$$

$$= 1-[P(A)+P(B)-P(A).P(B)] \quad [\because A \,\&\, B \text{ are independent}]$$

$$= 1-[0.5+0.8-(0.5\times 0.8)]$$

$$= 1-[1.3-0.4] = 1-0.9 = 0.1$$

⑨ Event A & B are such that $P(A+B)=\frac{3}{4}$, $P(AB)=\frac{1}{4}$ & $P(\bar{A})=\frac{2}{3}$ find $P(B)$.

Sol: $P(A)=1-P(\bar{A})=1-\frac{2}{3}=\frac{1}{3}$

$$P(A+B)=P(A)+P(B)-P(AB)$$

$$\frac{3}{4}=\frac{1}{3}+P(B)-\frac{1}{4}$$

$$\therefore P(B) = \frac{3}{4}-\frac{1}{3}+\frac{1}{4} = 1-\frac{1}{3} = \frac{2}{3}$$

⑩ A total of 36 members of a club play Tennis, 28 play squash, & 18 play badminton. Furthermore, 22 of the members play both tennis & squash 12 play both tennis & badminton, 9 play both squash & badminton, & 4 play all the 3 sports. How many members of this club play atleast one of these sports?

Sol:

$$P[\text{play atleast one of these sports}] = P(T\cup S\cup B)$$

$$= P(T)+P(S)+P(B)-P(T\cap S)-P(S\cap B)-P(T\cap B)+P(T\cap S\cap B)$$

$$= \frac{36+28+18-22-9-12+4}{N} = \frac{43}{N}$$

Hence 43 members play atleast one of these sports.

⑪ Out of $(2n+1)$ tickets consecutively numbered three are drawn at random. Find the probability that the numbers on them are in arithmetic progression.

Sol: $n(S) = (2n+1)C_3 = \frac{(2n+1)\,2n\,(2n-1)}{3!} = \frac{n(4n^2-1)}{3}$

$d = 1, 2, 3, \ldots, (n-1), n$ (Difference)

If $d=1$

$$\left.\begin{array}{l} 1, 2, 3 \\ 2, 3, 4 \\ \vdots \\ (2n-1)\,2n\,(2n+1) \end{array}\right\} \text{totally } (2n-1) \text{ cases}$$

If $d=2$

$$\left.\begin{array}{l} 1, 3, 5 \\ 2, 4, 6 \\ \vdots \\ (2n-3)(2n-1)(2n+1) \end{array}\right\} \text{totally } (2n-3) \text{ cases}$$

If $d=n-1$

$$\left.\begin{array}{l} 1, n, 2n-1 \\ 2, n+1, 2n \\ 3, n+2, 2n+1 \end{array}\right\} \text{totally } 3 \text{ cases}$$

If $d=n$ $\quad 1, n+1, 2n+1 \rightarrow$ totally 1 case

$\therefore n(A) = (2n-1) + (2n-3) + \cdots + 3 + 1 = \frac{n}{2}(1 + (2n-1)) = \frac{n}{2} \times 2n = n^2$

$\therefore P(A) = \frac{n(A)}{n(S)} = \frac{n^2}{n(4n^2-1)} \times 3 = \frac{3n}{4n^2-1}$

⑫ A can hit a target in 4 out of 5 shots & B can hit the target in 3 out of 4 shots. Find the probability that (i) the target being hit when both try. (ii) the target being hit by exactly one person.

Sol: $P(A) = \frac{4}{5}$, $P(B) = \frac{3}{4}$

(i) $P(A \cup B) = P(A) + P(B) - P(A \cap B) = P(A) + P(B) - P(A).P(B)$

$= \frac{4}{5} + \frac{3}{4} - \left(\frac{4}{5} \cdot \frac{3}{4}\right) = \frac{19}{20}$

(ii) P[The target being hit by exactly one person]

$= P[(A \cap \bar{B}) \cup (B \cap \bar{A})] = P[(A \cap \bar{B}) + (B \cap \bar{A})]$

$= P(A)\,P(\bar{B}) + P(B)\,P(\bar{A})$

$= P(A)(1 - P(B)) + P(B)(1 - P(A))$

$= \frac{4}{5}\left(1 - \frac{3}{4}\right) + \frac{3}{4}\left(1 - \frac{4}{5}\right)$

$$= \left(\frac{4}{5} \times \frac{1}{4}\right) + \left(\frac{3}{4} \times \frac{1}{5}\right)$$

$$= \frac{1}{5} + \frac{3}{20} = \frac{7}{20}$$

Marginal Probability:

A probability of only one event that takes place is called a marginal probability.

Joint Probability:

The probability of occurrence of both events A & B together, denoted by $P(A \cap B)$, is known as joint probability of A & B.

Conditional Probability:

The conditional probability of A given B is

$$P(A/B) = \frac{P(A \cap B)}{P(B)}$$ if $P(B) \neq 0$ & it is undefined otherwise.

Problems:

① When 2 dice are thrown (or a die is thrown twice). Let A be the event that the sum of the points on the faces is odd & B is the event that atleast one number is 2. Find the probabilities of the following:

① A ② B ③ $\bar{A}$ ④ $\bar{B}$ ⑤ $A \cap B$ ⑥ $A \cup B$ ⑦ $\bar{A} \cap B$ ⑧ $A \cap \bar{B}$ ⑨ $\bar{A} \cap \bar{B}$ ⑩ $\bar{A} \cup B$

⑪ $A \cup \bar{B}$ ⑫ $\bar{A} \cup \bar{B}$ ⑬ A/B ⑭ B/A.

Sol: The sample space is

$$S = \left\{ \begin{array}{l} (1,1),(1,2),(1,3),(1,4),(1,5),(1,6) \\ \vdots \\ (6,1),(6,2),(6,3),(6,4),(6,5),(6,6) \end{array} \right\}$$

$n(S) = 36$

$A = \{(1,2),(1,4),(1,6),(2,1),(2,3),(2,5),(3,2),(3,4),(3,6),$
$(4,1),(4,3),(4,5),(5,2),(5,4),(5,6),(6,1),(6,3),(6,5)\}$

$n(A) = 18$

$B = \{(1,2),(2,1),(2,2),(2,3),(2,4),(2,5),(2,6),(3,2),(4,2),(5,2),$
$(6,2)\}$

$n(B) = 11$

$A \cap B = \{(1,2),(2,1),(2,3),(2,5),(3,2),(5,2)\}$

$n(A\cap B) = 6$

① $P(A) = \frac{n(A)}{n(S)} = \frac{18}{36} = \frac{1}{2}$ ② $P(B) = \frac{n(B)}{n(S)} = \frac{11}{36}$

③ $P(\bar{A}) = 1 - P(A) = 1 - \frac{1}{2} = \frac{1}{2}$ ④ $P(\bar{B}) = 1 - P(B) = 1 - \frac{11}{36} = \frac{25}{36}$

⑤ $P(A\cap B) = \frac{n(A\cap B)}{n(S)} = \frac{6}{36} = \frac{1}{6}$

⑥ $P(A\cup B) = P(A) + P(B) - P(A\cap B) = \frac{1}{2} + \frac{11}{36} - \frac{1}{6} = \frac{23}{36}$

⑦ $P(\bar{A}\cap B) = P(B) - P(A\cap B) = \frac{11}{36} - \frac{1}{6} = \frac{5}{36}$

⑧ $P(A\cap \bar{B}) = P(A) - P(A\cap B) = \frac{1}{2} - \frac{1}{6} = \frac{1}{3}$

⑨ $P(\bar{A}\cap\bar{B}) = P(\overline{A\cup B}) = 1 - P(A\cup B) = 1 - \frac{23}{36} = \frac{13}{36}$

⑩ $P(\bar{A}\cup B) = P(\bar{A}) + P(B) - P(\bar{A}\cap B) = \frac{1}{2} + \frac{11}{36} - \frac{5}{36} = \frac{2}{3}$

⑪ $P(A\cup\bar{B}) = P(A) + P(\bar{B}) - P(A\cap\bar{B}) = \frac{1}{2} + \frac{25}{36} - \frac{1}{3} = \frac{31}{36}$

⑫ $P(\bar{A}\cup\bar{B}) = P(\bar{A}) + P(\bar{B}) - P(\bar{A}\cap\bar{B}) = \frac{1}{2} + \frac{25}{36} - \frac{13}{36} = \frac{5}{6}$

⑬ $P(A|B) = \frac{P(A\cap B)}{P(B)} = \frac{1/6}{11/36} = \frac{6}{11}$

⑭ $P(B|A) = \frac{P(A\cap B)}{P(A)} = \frac{1/6}{1/2} = \frac{1}{3}$

Baye's theorem: (Theorem of probability of cases)

Let $B_1, B_2, \ldots, B_n$ be an exhaustive & mutually exclusive random experiments & A be an event related to that B_i then

$$P(B_i|A) = \frac{P(B_i)\,P(A|B_i)}{\sum_{i=1}^{n} P(B_i)\,P(A|B_i)}$$

Proof: According to conditional probability

$$P(B_i|A) = \frac{P(B_i\cap A)}{P(A)} \quad \text{—①}$$

Using multiplication rule of probability

$$P(B_i \cap A) = P(B_i)\,P(A|B_i) \quad \text{---} ②$$

Using total probability theorem

$$P(A) = \sum_{i=1}^{n} P(B_i)\,P(A|B_i) \quad \text{---} ③$$

$$\therefore ① \Rightarrow P(B_i|A) = \frac{P(B_i)P(A|B_i)}{\sum_{i=1}^{n} P(B_i)P(A|B_i)} \quad \text{by } ② \,\&\, ③.$$

Problems:

① The contents of urns I, II, III are as follows:

Balls Urns	White	Black	Red
I	1	2	3
II	2	1	1
III	4	5	3

One urn is chosen at random & 2 balls are drawn. They happen to be white & red. What is the probability that they come from urns I, II & III ?

Sol: Let B_1, B_2, B_3 denote the events that the urns I, II, III are chosen respectively & let A be the event that the 2 balls taken from the selected urn are white & red.

Then $P(B_1) = P(B_2) = P(B_3) = \frac{1}{3}$.

$$P(A|B_1) = \frac{1C_1 \times 3C_1}{6C_2} = \frac{1}{5}$$

$$P(A|B_2) = \frac{2C_1 \times 1C_1}{4C_2} = \frac{1}{3}$$

$$P(A|B_3) = \frac{4C_1 \times 3C_1}{12C_2} = \frac{2}{11}$$

Baye's theorem: $P(B_i|A) = \dfrac{P(B_i)\,P(A|B_i)}{\sum_{i=1}^{n} P(B_i)\,P(A|B_i)}$

$$P(B_1|A) = \frac{P(B_1)P(A|B_1)}{\sum_{i=1}^{3} P(B_i)P(A|B_i)} = \frac{P(B_1)P(A|B_1)}{P(B_1)P(A|B_1)+P(B_2)P(A|B_2)+P(B_3)P(A|B_3)}$$

$$= \frac{1/3 \times 1/5}{(1/3 \times 1/5)+(1/3 \times 1/3)+(1/3 \times \frac{2}{11})} = \frac{1/15}{1/15+\frac{1}{9}+\frac{2}{33}} = \frac{33}{118}$$

$$P(B_2|A) = \frac{P(B_2)P(A|B_2)}{\sum_{i=1}^{3} P(B_i)P(A|B_i)} = \frac{1/3 \times 1/3}{1/15+1/9+2/33} = \frac{55}{118}$$

$$P(B_3|A) = 1-P(B_1|A)-P(B_2|A) = 1-\frac{33}{118}-\frac{55}{118} = \frac{15}{59}$$

② Companies B_1, B_2 & B_3 produce 30%, 45% & 25% of the cars respectively. It is known that 2%, 3% & 2% of these cars produced from are defective. (i) What is the probability that a car purchased is defective? (ii) If a car purchased is found to be defective, what is the probability that this car is produced by company B_1?

Sol: Let x be the event that the car purchased is defective.

$$P(B_1) = 30\% = \frac{30}{100} = 0.3$$

$$P(x/B_1) = 2\% = \frac{2}{100} = 0.02$$

$$P(B_2) = 45\% = \frac{45}{100} = 0.45$$

$$P(x/B_2) = 3\% = \frac{3}{100} = 0.03$$

$$P(B_3) = 25\% = \frac{25}{100} = 0.25$$

$$P(x/B_3) = 2\% = \frac{2}{100} = 0.02$$

(i) $P(x) = P(B_1)P(x/B_1) + P(B_2)P(x/B_2) + P(B_3)P(x/B_3)$

$= (0.3 \times 0.02) + (0.45 \times 0.03) + (0.25 \times 0.02)$

$= 0.0245$

(ii) $$P(B_1/x) = \frac{P(B_1)P(x/B_1)}{P(x)} = \frac{(0.3 \times 0.02)}{0.0245} = \frac{12}{49}$$

③ A given lot of Ic chips contains 2% defective chips. Each is tested before delivery. The tester itself is not totally reliable. Probability of tester says the chip is good when it is really good is 0.95 & the probability of tester says chip is defective when it is actually defective is 0.94. If a tested device is indicated to be defective, what is the probability that it is actually defective.

Sol:

E → Event of chip is actually good.

$\bar{E}$ → Event of chip is actually defective.

We know that $P(E)+P(\bar{E})=1$.

D → Event of tester says it is good.

$\bar{D}$ → Event of tester says it is defective.

Given: Lot of Ic Chips contains 2% defective chips.

(ie) $P(\bar{E}) = 2\% = \frac{2}{100} = 0.02$

$P(E) = 1 - P(\bar{E}) = 1 - 0.02 = 0.98$

Given: Probl. of tester says the chip is good when it is really good is 0.95.

(ie) $P(D/E) = 0.95$

$P(\bar{D}/E) = 1 - P(D/E) = 1 - 0.95 = 0.05$

Given: Probl. of the tester says the chip is defective when it is actually defective is 0.94.

(ie) $P(\bar{D}/\bar{E}) = 0.94$

To find: The probl. of actually defective

(ie) $P(\bar{E}/\bar{D})$

By Baye's theorem,

$$P(\bar{E}/\bar{D}) = \frac{P(\bar{D}/\bar{E}) \cdot P(\bar{E})}{P(\bar{D}/\bar{E}) \cdot P(\bar{E}) + P(\bar{D}/E) \cdot P(E)}$$

$$= \frac{0.94 \times 0.02}{(0.94 \times 0.02) + (0.05 \times 0.98)} = 0.2773$$

④ A bag contains 3 black & 4 white balls. Two balls are drawn at random one at a time without replacement.

(i) What is the probability that the second ball drawn is white?

(ii) What is the conditional probability that the first ball drawn is white if the second ball is known to be white?

Sol: Given: 3 black balls, 4 white balls

Total no. of balls = 3+4 = 7.

Let A → the first ball drawn is white.

B → the second ball drawn is white.

Second ball is white; it can happen in two mutually exclusive ways.

(1) First ball is white & second is white.

(2) First ball is black & second is white.

(i) $P(B) = P(1) + P(2) = \left(\frac{4}{7} \times \frac{3}{6}\right) + \left(\frac{3}{7} \times \frac{4}{6}\right)$

$$= \frac{12}{42} + \frac{12}{42} = \frac{24}{42} = \frac{4}{7}$$

(ii) $P(A|B) = \frac{P(A \cap B)}{P(B)}$

$A \cap B$ = first ball was white & second ball is also white.

$P(A \cap B) = \frac{4}{7} \times \frac{3}{6} = \frac{2}{7}$

$\therefore P(A|B) = \frac{2/7}{4/7} = \frac{1}{2}$

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⑤ A consulting firm rents cars from 3 rental agencies in the following manner: 20% from agency D, 20% from agency E one 60% from agency F. If 10% cars from D, 12% of the cars from E & 4% of the cars from F have bad tyres. What is the probability that the firm will get a car with bad tyres? Find the firm will get a car with bad tyres? Find the prob. that a car with bad tyres is rented from agency F.

Sol: Let A be the event that the car has bad tyres.

Given: $P(D) = 20\% = 0.2$

$P(A/D) = 10\% = 0.1$

$P(E) = 20\% = 0.2$

$P(A/E) = 12\% = 0.12$

$P(F) = 60\% = 0.6$

$P(A/F) = 4\% = 0.04$

$$P(A) = P(D)\,P(A/D) + P(E)\,P(A/E) + P(F)\,P(A/F)$$

$$= (0.2 \times 0.1) + (0.2 \times 0.12) + (0.6 \times 0.04)$$

$$= 0.068$$

$$P(F/A) = \frac{P(A \cap F)}{P(A)} = \frac{P(F)\,P(A/F)}{P(A)}$$

$$= \frac{0.6 \times 0.04}{0.068} = \frac{6}{17}$$

Random Variables:

Introduction:

Random variable is a real valued function which maps the numerical or non-numerical sample space (domain) of the random experiment to real values (codomain or range).

Defn.:

Let S be the sample space of the random experiment. Random
variable is a function whose domain is the set of outcomes $w \in S$ & whose
range is R, the real line. The random variable assigns a real value $X(w)$
such that (i) The set $\{w : X(w) \leq x\}$ is an event for every $x \in R$, for
which a probability is defined. This condition is called as measurability.

(ii) $P(X = \infty) = P(X = -\infty) = 0$

(iii) For every set $A \subset S$ there corresponds a set $T \subset R$ called the image of A. Also for every Borel set $T \subset R$ there exists in S the inverse image $X^{-1}(T)$.

The random variable is denoted by upper case letter (say, X) & the real values that it takes, are denoted by lower case letters (say, x).

Defn.:

The set of values which the random variable X takes is called as spectrum of the random variable.

E.g: Consider an experiment of tossing an unbiased coin twice. Let the random variable X denote the no. of heads turning up. The outcome of the experiments are HH, HT, TH & TT. Hence X can take values 2, 1, 1, 0. These values are called the spectrum of the random variable X.

Defn.:

A coin is tossed once. I denotes the appearance of tail. Describe the random variable $I = \begin{cases} 0, & \text{if head appears} \\ 1, & \text{if tail appears} \end{cases}$. I is called indicator function. (I equals 1 or 0 depending on whether or not the event occurs).

Types of Random variable:

A random variable can be classified as real random variable & complex random variable. Real random variable is one which takes on real values. A complex random variable Z is defined in terms of real random variables X & Y as $Z = X + iY$ where $i = \sqrt{-1}$.

A real random variable may be further classified as discrete random variable, continuous random variable & mixed random variable. These random variables may be further classified as one-dimensional random variable & multi-dimensional random variable.

Discrete Random variable:

A random variable X is discrete if it assumes only discrete values. The sample space (domain) of discrete random variables may be discrete, continuous or a mixture of discrete & continuous pts, but the co-domain is only discrete.

Eg:- ① The mark obtained by a student in an examination. Its possible values are 0, 40, 85 or 100.

② The no. of students who are absent for a particular period.

Continuous Random variable:

A random variable X is said to be a continuous random variable if it takes all possible values between certain limits or in an interval which may be finite or infinite. The sample space of the continuous random variable must be continuous & cannot be discrete or mixed as otherwise it will violate the single-valued condition.

Eg:- ① The density of milk taken for testing at a farm.

② The operating time between two failures of a computer.

Mixed Random variable:

A random variable which can take both discrete & continuous values is called a mixed random variable.

One Dimensional Random variable:

If a random variable X takes on single value corresponding to each outcome of the experiment, then the random variable is called one dimensional random variable. One dimensional random variable is also called as Scalar-valued random variable.

Eg:- In coin tossing experiment if we assume the random variable to be appearance of tail, then the sample space is {H, T} & the random variable is {0, 1}, which is an one-dimensional random variable.

Defn.:

A funƒ. which assigns a vector to each pt of sample space is called a vector random variable or a random vector.

E.g:- Consider an experiment of throwing a die & tossing a coin together. Here the sample space is two-dimensional, namely {(H,1),(H,2),(H,3),(H,4),(H,5),(H,6),(T,1),(T,2),(T,3),(T,4),(T,5),(T,6)}. We can define two-dimensional random variables over this sample space. For example if we define the two-dimensional random variable as appearance of head & an even no/., then the values of the random variable are (1,0),(1,1),(1,0),(1,1),(1,0),(1,1),(0,0),(0,1),(0,0),(0,1),(0,0),(0,1).

Defn.: Distribution funƒ. of X (or) Cumulative distribution of X:

The distribution funƒ. of a random variable X defined in $(-\infty,\infty)$ is given by $F(x) = P(X \leq x)$

Probability mass funƒ. (or) Probability funƒ.:

Let X be a one dimensional discrete R.V. which takes the values $x_1, x_2, x_3, \ldots$ $P(X=x_i) = P(x_i) = P_i$ called the probability of x_i. The nos/. $P_i = P(x_i)$ satisfies the following conditions

(i) $P(x_i) \geq 0$ (ii) $\sum_{i=1}^{\infty} P(x_i) = 1$

The funƒ. $P(x)$ satisfying the above two conditions is called the probability mass funƒ. (or) probability funƒ.. The set $\{x_i, p_i\}$ is called the probability distribution of the R.V. X.

Results:

① $P(X \leq \infty) = 1$ ② $P(X \leq -\infty) = 0$

③ If $x_1 \leq x_2$ then $P(X=x_1) \leq P(X=x_2)$

④ $P(X>x) = 1 - P(X \leq x)$ ⑤ $P(X \leq x) = 1 - P(X>x)$

Expected value of X of a discrete random variable X:-

Let X be a discrete random variable assuming values $x_1, x_2, \ldots, x_n$ with corresponding probabilities $P_1, P_2, \ldots, P_n$. Then $E(X) = \sum_i x_i P(x_i)$

is called the expected value of X. E(X) is also called the mean or the expectation of X.

Variance of X:

$$Var(x) = E[x^2] - [E(x)]^2$$

The quantity $\sqrt{Var(x)}$ is called the standard deviation of X.

Formulae:

① $\sum_{i=1}^{n} P(x_i) = 1$

② $F(x) = P(X \le x)$ (ie) E.g: $P(X \le 4) = F(4)$, $P(X \le 5) = F(5)$. $F(0) = P(0)$

$F(1) = P(0) + P(1)$

$F(2) = P(0) + P(1) + P(2) = F(1) + P(2)$

$F(3) = P(0) + P(1) + P(2) + P(3) = F(2) + P(3)$

③ $P(1) = F(1) - F(0)$

$P(2) = F(2) - F(1)$

$P(3) = F(3) - F(2)$

④ Mean $= E(X) = \sum_i x_i P(x_i)$

⑤ $E[x^2] = \sum_i x_i^2 P(x_i)$

⑥ Variance $= Var[x] = E[x^2] - [E(x)]^2$

⑦ $E[ax+b] = aE[x] + b$

⑧ $Var[ax \pm b] = a^2 Var(x)$

Problems:

① For the following probability distribution (i) Find the distribution fun/. of X. (ii) what is the smallest value of x for which $P(X \le x) > 0.5$.

x :	0	1	2
P(x):	1/4	2/4	1/4

Sol: (i) The distribution fun/. of X is given by $F(x) = P(X \le x)$.

x	$F(x) = P(X \le x)$
0	$F(0) = P(X \le 0) = P(X = 0) = 1/4$
1	$F(1) = P(X \le 1) = P(X=0) + P(X=1) = 1/4 + 2/4 = 3/4$
2	$F(2) = P(X \le 2) = \frac{1}{4} + \frac{2}{4} + \frac{1}{4} = 1$

(ii) The smallest value of x for which $P(X \leq x) > 0.5$ is 1.

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② Obtain the probability fun. (or) probability distribution from the following distribution fun..

x :	0	1	2	3
$F(x)$:	0.1	0.4	0.9	1

Sol: Probability fun.:

x	$P(x)$
0	$F(0) = P(0) = 0.1$
1	$P(1) = F(1) - F(0) = 0.4 - 0.1 = 0.3$
2	$P(2) = F(2) - F(1) = 0.9 - 0.4 = 0.5$
3	$P(3) = F(3) - F(2) = 1 - 0.9 = 0.1$

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③ When a die is thrown, X denotes the no. that turns up. Find $E(X)$, $E(X^2)$ & $Var(X)$.

Sol: X is a discrete random variable taking values 1, 2, 3, 4, 5, 6 & with probability $\frac{1}{6}$ for each.

x :	1	2	3	4	5	6
$P(x)$:	$1/6$	$1/6$	$1/6$	$1/6$	$1/6$	$1/6$

$$E(X) = \sum_i x_i P(x_i)$$

$$= \left(1 \times \frac{1}{6}\right) + \left(2 \times \frac{1}{6}\right) + \left(3 \times \frac{1}{6}\right) + \left(4 \times \frac{1}{6}\right) + \left(5 \times \frac{1}{6}\right) + \left(6 \times \frac{1}{6}\right)$$

$$= \frac{21}{6} = \frac{7}{2}$$

$$E(X^2) = \sum_i x_i^2 P(x_i) = \frac{1}{6}\left[1 + 4 + 9 + 16 + 25 + 36\right] = \frac{91}{6}$$

$$Var(X) = E[X^2] - [E(X)]^2 = \frac{91}{6} - \left(\frac{7}{2}\right)^2 = \frac{91}{6} - \frac{49}{4} = \frac{182 - 147}{12} = \frac{35}{12}$$

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④ A random variable X has the following probability fun.:

X :	0	1	2	3	4	5	6	7
P(X) :	0	k	$2k$	$2k$	$3k$	k^2	$2k^2$	$7k^2+k$

(a) Find k. (b) Evaluate $P[X < 6]$, $P[X \geq 6]$

(c) If $P[X \leq c] > \frac{1}{2}$ find the minimum value of c.

Sol: (a) Since $\sum_i P(x_i) = 1$

(ie) $0 + k + 2k + 2k + 3k + k^2 + 2k^2 + 7k^2 + k = 1 \Rightarrow 10k^2 + 9k - 1 = 0$

$$\Rightarrow (10k - 1)(10k + 10) = 0$$

$$\Rightarrow k = \frac{1}{\ }, -1$$

Since $P(X) \geq 0$ then we have $k = 1/10$.

(b) $P[X \geq 6] = P[X=6] + P[X=7]$

$$= 2k^2 + 7k^2 + k = 9k^2 + k = 9\left(\frac{1}{10}\right)^2 + \frac{1}{10} = \frac{9}{100} + \frac{1}{10} = \frac{9+10}{100} = \frac{19}{100}$$

$$P[X<6] = 1 - P[X \geq 6] = 1 - \frac{19}{100} = \frac{100-19}{100} = \frac{81}{100}$$

(c)

x	$P(x)$	$P[X \leq x]$
0	0	$P[X \leq 0] = P(x=0) = 0$
1	k	$P[X \leq 1] = 0 + k = k = 1/10$
2	$2k$	$P[X \leq 2] = 3k = 3/10$
3	$2k$	$P[X \leq 3] = 5k = 1/2$
4	$3k$	$P[X \leq 4] = 8k = 4/5$
~~5~~	~~k^2~~	~~$P[X \leq 5] = 8k + k^2 = 8/10 + 1/100 = 81/100$~~
~~6~~	~~$2k^2$~~	~~$P[X \leq 6] =$~~
~~7~~	~~$7k^2 + k$~~	

$P[X \leq 4] = \frac{4}{5} > 1/2$. Hence 4 is the minimum value of c.

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⑤ If X has the distribution funˢ. $F[x] = \begin{cases} 0 & \text{for } x < 1 \\ 1/3 & \text{for } 1 \leq x < 4 \\ 1/2 & \text{for } 4 \leq x < 6 \\ 5/6 & \text{for } 6 \leq x < 10 \\ 1 & \text{for } x \geq 10 \end{cases}$

Find (i) The probability distribution of X.

(ii) $P(2 < X < 6)$

(iii) Mean of X

(iv) Variance of X.

Sol: (i) Probability distribution of X:

x :	0	1	4	6	10
$P(x)$:	0	$1/3$	$1/6$	$1/3$	$1/6$

(ii) $P(2 < X < 6) = P(X=4) = \frac{1}{6}$

(iii) Mean of $X = E[X] = \sum_i x_i P(x_i)$

$$= (0 \times 0) + \left(1 \times \frac{1}{3}\right) + \left(4 \times \frac{1}{6}\right) + \left(6 \times \frac{1}{3}\right) + \left(10 \times \frac{1}{6}\right)$$

$$= \frac{1}{3} + \frac{2}{3} + 2 + \frac{5}{3} = \frac{8}{3} + 2 = \frac{14}{3}$$

(iv) $E[X^2] = \sum_i x_i^2 P(x_i) = (0 \times 0) + \left(1 \times \frac{1}{3}\right) + \left(16 \times \frac{1}{6}\right) + \left(36 \times \frac{1}{3}\right) + \left(100 \times \frac{1}{6}\right)$

$$= 0 + \frac{1}{3} + \frac{8}{3} + 12 + \frac{50}{3} = \frac{59}{3} + 12 = \frac{95}{3}$$

$$Var(X) = E[X^2] - [E(X)]^2 = \frac{95}{3} - \left(\frac{14}{3}\right)^2 = \frac{95}{3} - \frac{196}{9} = \frac{285-196}{9}$$

$$= \frac{89}{9}$$

⑥ If $Var(x)=4$, find $Var(3x+8)$, where X is a random variable.

Sol: WKT $Var(ax+b)=a^2 Var(x)$

$$Var(3x+8)=9\,Var(x)=9\times4=36$$

⑦ X & Y are independent random variables with variance 2 & 3. Find the variance of $3X+4Y$.

Sol: Given $Var(x)=2$ & $Var(Y)=3$

$$Var(3x+4Y)=3^2\,Var(x)+4^2\,Var(Y)$$

$$=(9\times2)+(16\times3)=66$$

⑧ If x be a random variable with $E(x)=1$ & $E[x(x-1)]=4$. Find $Var\,X$, $Var(2-3X)$ & $Var\left[\frac{X}{2}\right]$.

Sol: Given $E(x)=1$ & $E[x(x-1)]=4$

$$E[x(x-1)]=E[x^2-x]=E[x^2]-E[x]=E(x^2)-1$$

$$\Rightarrow 4=E(x^2)-1 \Rightarrow E[x^2]=5$$

$$Var(x)=E(x^2)-[E(x)]^2=5-1=4$$

$$Var(2-3x)=Var(2+(-3)x)=(-3)^2\,Var(x)=9\times4=36$$

$$Var\left(\frac{X}{2}\right)=\left(\frac{1}{2}\right)^2 Var(x)=\frac{1}{4}\times4=1$$

⑨ The probability funf. of an infinite discrete distribution is given by $P[X=j]=\frac{1}{2^j}$, $j=1,2,\dots,\infty$. Find the mean & variance of the distribution. Also find $P[x \text{ is even}]$, $P[x\geq5]$ & $P[x \text{ is divisible by } 3]$.

Sol: Given $P[x=j]=\frac{1}{2^j}$

$j:$	1	2	3	4	…
$P(X=j):$	$1/2$	$1/2^2$	$1/2^3$	$1/2^4$	…

$$E[x]=\sum_{j=1}^{\infty} x_j P(x_j)=(1)\left(\frac{1}{2}\right)+(2)\left(\frac{1}{2}\right)^2+(3)\left(\frac{1}{2}\right)^3+\dots$$

$$=\frac{1}{2}\left[1+2\left(\frac{1}{2}\right)+3\left(\frac{1}{2}\right)^2+\dots\right]$$

$$=\frac{1}{2}\left[1-\frac{1}{2}\right]^{-2} \quad (\because (1-x)^{-2}=1+2x+3x^2+4x^3+\dots)$$

$$=\frac{4}{2}=2$$

$$E[x^2]=\sum_{j=1}^{\infty} x_j^2 P(x_j)=\sum_{j=1}^{\infty}(x_j^2+x_j-x_j)P(x_j)=\sum_{j=1}^{\infty}(x_j^2+x_j)P(x_j)-\sum_{j=1}^{\infty}x_jP(x_j)$$

$$=\sum_{j=1}^{\infty} x_j(x_j+1)P(x_j)-2=\left[(1)(2)\left(\frac{1}{2}\right)+(2)(3)\left(\frac{1}{2}\right)^2+(3)(4)\left(\frac{1}{2}\right)^3+\dots\right]-2$$

$$=\frac{1}{2}\left[1.2+2.3.\left(\frac{1}{2}\right)+3.4.\left(\frac{1}{2}\right)^2+\dots\right]-2$$

$$=\frac{2}{2}\left[1+3\left(\frac{1}{2}\right)+6\left(\frac{1}{2}\right)^2+\dots\right]-2$$

$= \left[1-\frac{1}{2}\right]^{-3} - 2 \qquad \left(\because (1-x)^{-3} = 1+3x+6x^2+10x^3+\dots\right)$

$= 8-2 = 6$

$Var(X) = E[X^2] - [E(X)]^2 = 6-4 = 2$

$P[X \text{ is even}] = P[X=2] + P[X=4] + \dots$

$= \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^4 + \left(\frac{1}{2}\right)^6 + \dots$

$= \frac{1}{4} + \left(\frac{1}{4}\right)^2 + \left(\frac{1}{4}\right)^3 + \dots \qquad \left[(1-x)^{-1} = 1+x+x^2+\dots\right]$

$= \left[1-\frac{1}{4}\right]^{-1} - 1 = \frac{4}{3} - 1 = \frac{1}{3}$

$P[X \geq 5] = P[X=5] + P[X=6] + \dots$

$= \left(\frac{1}{2}\right)^5 + \left(\frac{1}{2}\right)^6 + \dots = \left(\frac{1}{2}\right)^5 \left[1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \dots\right]$

$= \left(\frac{1}{2}\right)^5 \left[1-\frac{1}{2}\right]^{-1} = \left(\frac{1}{2}\right)^5 \times 2 = \frac{1}{16}$

$P[X \text{ is divisible by } 3] = P[X=3] + P[X=6] + \dots$

$= \left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^6 + \dots$

$= \frac{1}{8} + \left(\frac{1}{8}\right)^2 + \left(\frac{1}{8}\right)^3 + \dots$

$= \left[1-\frac{1}{8}\right]^{-1} - 1 = \frac{8}{7} - 1 = \frac{1}{7}$

Continuous Random Variables:

If X is a continuous random variable for any x_1 & x_2

$P(x_1 \leq X \leq x_2) = P(x_1 < X \leq x_2) = P(x_1 \leq X < x_2) = P(x_1 < X < x_2)$.

Probability density fun.:

For a continuous random variable X, a probability density fun. is a fun. such that (i) $f(x) \geq 0$ (ii) $\int_{-\infty}^{\infty} f(x)\,dx = 1$

(iii) $P(a \leq X \leq b) = \int_a^b f(x)\,dx$ = area under $f(x)$ from a to b for any a & b.

Cumulative distribution fun.:

The cumulative distribution fun. of a continuous random variable X is $F(x) = P(X \leq x) = \int_{-\infty}^{x} f(t)\,dt$ for $-\infty < x < \infty$.

Result: (i) $f(x) = \frac{d}{dx} F[x]$

(ii) Mean $= \mu = E(X) = \int_{-\infty}^{\infty} x f(x) dx$

(iii) $E[g(x)] = \int_{-\infty}^{\infty} g(x) f(x) dx$

(iv) $Var(x) = \sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 f(x) dx = \int_{-\infty}^{\infty} x^2 f(x) dx - \mu^2$

$$= E(X^2) - [E(X)]^2$$

(v) Standard deviation of X $= \sigma = \sqrt{Var(X)}$.

Problems:

① Given that the p.d.f. of a R.V. X is $f(x) = kx$, $0 < x < 1$ find k & $P(x > 0.5)$.

Sol: WKT $\int_{-\infty}^{\infty} f(x) dx = 1 \Rightarrow \int_{0}^{1} kx\, dx = 1$

$$\Rightarrow k \left(\frac{x^2}{2}\right)_0^1 = 1 \Rightarrow \frac{k}{2}(1-0) = 1 \Rightarrow k = 2$$

$$P(x > 0.5) = \int_{0.5}^{\infty} f(x) dx = \int_{0.5}^{1} kx\, dx = 2\left(\frac{x^2}{2}\right)_{0.5}^{1} = (1 - 0.25) = 0.75$$

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② A continuous random variable x has p.d.f. given by $f(x) = 3x^2$, $0 \le x \le 1$. Find k such that $P(x > k) = 0.5$.

Sol: Given $f(x) = 3x^2$, $0 \le x \le 1$

$$P(x > k) = 0.5 \Rightarrow \int_{k}^{\infty} f(x) dx = 0.5 \Rightarrow \int_{k}^{1} 3x^2 dx = 0.5$$

$$\Rightarrow 3\left(\frac{x^3}{3}\right)_k^1 = 0.5 \Rightarrow 1 - k^3 = 0.5 \Rightarrow k^3 = 0.5$$

$$\Rightarrow k = (0.5)^{1/3} = 0.7937$$

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③ The p.d.f. of a continuous R.V. X is $f(x) = ke^{-|x|}$. Find k & the F[x].

Sol: WKT $\int_{-\infty}^{\infty} f(x) dx = 1 \Rightarrow \int_{-\infty}^{\infty} ke^{-|x|} dx = 1$

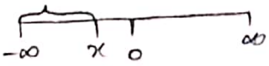
$$\Rightarrow 2k \int_{0}^{\infty} e^{-x} dx = 1 \Rightarrow 2k\left(\frac{e^{-x}}{-1}\right)_0^{\infty} = 1$$

$$\Rightarrow -2k(0-1) = 1 \Rightarrow 2k = 1 \Rightarrow k = 1/2$$

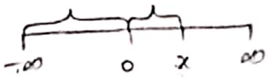
$$F(x)=\int_{-\infty}^{x} f(x)\,dx$$

Given $f(x)=ke^{-|x|}=\begin{cases} ke^{x}, & -\infty<x\leq 0 \\ ke^{-x}, & 0<x<\infty \end{cases}=\begin{cases} \frac{1}{2}e^{x}, & -\infty<x\leq 0 \\ \frac{1}{2}e^{-x}, & 0<x<\infty \end{cases}$

For $x\leq 0$, $F(x)=\int_{-\infty}^{x}\frac{1}{2}e^{x}dx=\frac{1}{2}\left(e^{x}\right)_{-\infty}^{x}=\frac{1}{2}\left(e^{x}-0\right)=\frac{e^{x}}{2}$



For $x>0$, $F(x)=\int_{-\infty}^{0}\frac{1}{2}e^{x}dx+\int_{0}^{x}\frac{1}{2}e^{-x}dx$



$$=\frac{1}{2}\left(e^{x}\right)_{-\infty}^{0}+\frac{1}{-2}\left(e^{-x}\right)_{0}^{x}$$

$$=\frac{1}{2}(1-0)+\frac{1}{-2}\left(e^{-x}-1\right)=\frac{-1}{2}e^{-x}+1=1-\frac{1}{2}e^{-x}$$

$$\therefore F(x)=\begin{cases} \frac{e^{x}}{2}, & x\leq 0 \\ 1-\frac{1}{2}e^{-x}, & x>0 \end{cases}$$

④ If X is a continuous R.V. with p.d.f. $f(x)=\begin{cases} x, & 0\leq x<1 \\ \frac{3}{2}(x-1)^2, & 1\leq x<2 \\ 0, & \text{otherwise} \end{cases}$ find the cumulative distribution funˡ. $F[x]$ of X & use it to find $P\left[\frac{3}{2}<X<\frac{5}{2}\right]$.

Sol: (i) If $x<0$ then $F(x)=\int_{-\infty}^{x}f(x)dx=\int_{-\infty}^{}0.dx=0$

(ii) If $0\leq x<1$ then $F(x)=\int_{-\infty}^{x}f(x)dx=\int_{0}^{x}x\,dx=\left(\frac{x^2}{2}\right)_0^x=\frac{x^2}{2}$

(iii) If $1\leq x<2$ then $F(x)=\int_{-\infty}^{x}f(x)dx\cancel{=\int_{1}^{x}f(x)dx=\frac{3}{2}\int_{1}^{x}(x-1)^2dx}$

$$\cancel{=\frac{3}{2}\left(\frac{(x-1)^3}{3}\right)_1^x=\frac{1}{2}(x-1)^3}$$

$$F(x)=\int_{\cancel{-\infty}0}^{1}f(x)dx+\int_{1}^{x}f(x)dx=\int_{\cancel{-\infty}0}^{1}x\,dx+\int_{1}^{x}\frac{3}{2}(x-1)^2dx$$

$$=\left(\frac{x^2}{2}\right)_0^1+\frac{3}{2}\left[\frac{(x-1)^3}{3}\right]_1^x=\frac{1}{2}+\frac{1}{2}(x-1)^3$$

(iv) If $x\geq 2$ then $F(x)=\int_{-\infty}^{x}f(x)dx=\int_{0}^{1}x\,dx+\int_{1}^{2}\frac{3}{2}(x-1)^2dx+\int_{2}^{x}0.dx$

$$=\left(\frac{x^2}{2}\right)_0^1+\frac{3}{2}\left(\frac{(x-1)^3}{3}\right)_1^2=\frac{1}{2}+\frac{1}{2}[1]=1$$

$$\therefore F(x)=\begin{cases} 0, & x<0 \\ \frac{x^2}{2}, & 0\leq x<1 \\ \frac{1}{2}+\frac{1}{2}(x-1)^3, & 1\leq x<2\ ;\ 1, \ x\geq 2 \end{cases}$$

$$P\left[\frac{3}{2} < x < \frac{5}{2}\right] = F\left[\frac{5}{2}\right] - F\left[\frac{3}{2}\right]$$

$$= 1 - \left(\frac{1}{2} + \frac{1}{2}\left(\frac{3}{2} - 1\right)^3\right) = 1 - \left(\frac{1}{2} + \frac{1}{16}\right)$$

$$= 1 - \frac{9}{16} = \frac{7}{16}$$

⑤ A continuous random variable X has the distribution fun/. $F[x] = \begin{cases} 0, & x \le 1 \\ K(x-1)^4, & 1 < x \le 3 \\ 0, & x > 3 \end{cases}$ find K, probability density fun/. $f(x)$, $P[X < 2]$.

Sol: WKT $f(x) = \frac{d}{dx} F[x]$

$$\therefore f(x) = \begin{cases} 0, & x \le 1 \\ 4K(x-1)^3, & 1 < x \le 3 \\ 0, & x > 3 \end{cases}$$

WKT $\int_{-\infty}^{\infty} f(x)\,dx = 1 \Rightarrow \int_1^3 4K(x-1)^3\,dx = 1 \Rightarrow 4K\left(\frac{(x-1)^4}{4}\right)_1^3 = 1$

$$\Rightarrow K(2)^4 = 1 \Rightarrow K = \frac{1}{16}$$

$$P[X < 2] = F(2) = K(2-1)^4 = \frac{1}{16}$$

⑥ Is the fun/. defined as follows a density fun/.? $f(x) = \begin{cases} 0 & \text{for } x < 2 \\ \frac{1}{18}(3+2x) & \text{for } 2 \le x \le 4 \\ 0 & \text{for } x > 4 \end{cases}$

Sol: Condition for p.d.f. is $\int_{-\infty}^{\infty} f(x)\,dx = 1$.

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{2} 0.dx + \int_2^4 \frac{1}{18}(3+2x)\,dx + \int_4^{\infty} 0.dx$$

$$= \frac{1}{18}\left[3x + 2\frac{x^2}{2}\right]_2^4 = \frac{1}{18}\left(3x + x^2\right)_2^4 = \frac{1}{18}\left[12 + 16 - 6 - 4\right] = 1$$

Hence the given fun/. is density fun/..

⑦ If the density fun/. of a continuous R.V. X is given by $f(x) = \begin{cases} ax, & 0 \le x \le 1 \\ a, & 1 \le x \le 2 \\ 3a - ax, & 2 \le x \le 3 \\ 0, & \text{otherwise} \end{cases}$

(i) Find the value of a. (ii) the cumulative distribution fun/. of X.

(iii) If x_1, x_2 & x_3 are 3 independent observations of X.

What is the probability that exactly one of these 3 is greater than 1.5?

Sol: (i) WKT, $\int_{-\infty}^{\infty} f(x)\,dx = 1 \Rightarrow \int_0^1 ax\,dx + \int_1^2 a\,dx + \int_2^3 (3a - ax)\,dx = 1$

$$\Rightarrow a\left(\frac{x^2}{2}\right)_0^1 + a(x)_1^2 + \left(3ax - a\frac{x^2}{2}\right)_2^3 = 1$$

$$\Rightarrow \frac{a}{2}(1) + a(1) + \left(9a - \frac{9a}{2} - 6a + 2a\right) = 1$$

$$\Rightarrow \frac{a}{2} + a + 3a - \frac{5a}{2} = 1 \Rightarrow 2a = 1 \Rightarrow a = \frac{1}{2}$$

(ii) If $x < 0$ then $F(x) = 0$

If $0 \le x \le 1$ then $F[x] = \int_0^x ax\,dx = \frac{1}{2}\left(\frac{x^2}{2}\right)_0^x = \frac{1}{4}x^2$

If $1 \le x \le 2$ then $F(x) = \int_{-\infty}^{x} f(x)\,dx = \int_0^1 ax\,dx + \int_1^x a\,dx$

$$= \frac{1}{2}\left(\frac{x^2}{2}\right)_0^1 + \frac{1}{2}(x)_1^x = \frac{1}{4} + \frac{1}{2}(x-1)$$

$$= \frac{x}{2} + \frac{1}{4} - \frac{1}{2} = \frac{x}{2} - \frac{1}{4}$$

If $2 \le x \le 3$ then $F(x) = \int_0^1 ax\,dx + \int_1^2 a\,dx + \int_2^x (3a - ax)\,dx$

$$F(x) = \frac{1}{2}\left(\frac{x^2}{2}\right)_0^1 + a(x)_1^2 + \left[3ax - a\frac{x^2}{2}\right]_2^x$$

$$= \frac{1}{4} + \frac{1}{2} + \frac{3x}{2} - \frac{x^2}{4} - 3 + 1 = \frac{3}{4} - 2 + \frac{3x}{2} - \frac{x^2}{4}$$

$$= \frac{3x}{2} - \frac{x^2}{4} - \frac{5}{4}$$

If $x > 3$ then $F(x) = \int_0^1 \frac{1}{2}x\,dx + \int_1^2 \frac{1}{2}\,dx + \int_2^3 \left(\frac{3}{2} - \frac{x}{2}\right)dx + \int_3^x 0.dx$

$$= \frac{1}{4} + \frac{1}{2} + \left(\frac{3}{2}x - \frac{x^2}{4}\right)_2^3 = \frac{1}{4} + \frac{1}{2} + \frac{9}{2} - \frac{9}{4} - 3 + 1$$

$$\cancel{= \frac{-26}{4} + \frac{10}{2} = \frac{-13}{2} + 5 = \frac{-3}{2}} = -2 + 5 - 2 = 1$$

$$\therefore F(x) = \begin{cases} 0 & , x < 0 \\ \frac{x^2}{4} & , 0 \le x \le 1 \\ \frac{x}{2} - \frac{1}{4} & , 1 \le x \le 2 \\ \frac{3x}{2} - \frac{x^2}{4} - \frac{5}{4} & , 2 \le x \le 3 \\ 1 & , x > 3 \end{cases}$$

(iii) $P(X > 1.5) = \int_{1.5}^{3} f(x)\,dx = \int_{1.5}^{2} \frac{1}{2}\,dx + \int_2^3 \left(\frac{3}{2} - \frac{x}{2}\right)dx$

$$= \frac{1}{2}(x)_{1.5}^{2} + \left(\frac{3}{2}x - \frac{x^2}{4}\right)_2^3$$

$$= \frac{1}{2}(2 - 1.5) + \frac{9}{2} - \frac{9}{4} - 3 + 1 = \frac{1}{4} + \frac{9}{2} - \frac{9}{4} - 2 = \frac{9}{2} - 4 = \frac{1}{2}$$

Choosing an X & observing its value can be considered as a trial & $X > 1.5$ can be considered as a success. $\therefore p = \frac{1}{2}$, $q = \frac{1}{2}$

As we choose 3 independent observation of X, $n = 3$.

By Bernoulli's thm., P(exactly one value > 1.5) = P(1 success)

$$= {}^{n}C_x\, p^x q^{n-x} = 3C_1\left(\frac{1}{2}\right)^1\left(\frac{1}{2}\right)^2 = \frac{3}{8}$$

⑧ A coin is tossed an infinite no. of times. If the probability of a head in a single toss is p, find the probability that kth head is obtained at the nth tossing but not earlier, with $q=1-p$.

Sol: Given (i) A coin is tossed an infinite no. of times.

(ii) The probability of a head in a single toss is p.

(iii) $q=1-p$.

k heads should be obtained at the nth tossing, but not earlier.

∴ (k-1) heads must be obtained in the first (n-1) tosses & 1 head at the nth toss.

Required probability = P[(k-1) heads in (n-1) tosses] × P[1 head in one toss]

$$= \left[(n-1)C_{(k-1)}\, p^{k-1} q^{n-k}\right] \times p$$

$$= (n-1)C_{(k-1)}\, p^{k} q^{n-k}$$

⑨ The sales of a convenience store on a randomly selected day are x thousand dollars, where X is a random variable with a distribution funl. of the following form:

$$F(x) = \begin{cases} 0 & , x<0 \\ x^2/2 & , 0\le x<1 \\ k(4x-x^2) & , 1\le x<2 \\ 1 & , x\ge 2 \end{cases}$$

Suppose that this convenience store's total sales on any given day are less than \$ 2000.

(i) Find the value of k.

(ii) Let A & B be the events that tomorrow the store's total sales are between 500 & 1500 dollars, & over 1000 dollars respectively. Find P(A) & P(B).

(iii) Are A & B independent events?

Sol: WKT $f(x) = \frac{d}{dx}F(x) = \begin{cases} 0 & , x<0 \\ x & , 0\le x<1 \\ k(4-2x) & , 1\le x<2 \\ 0 & , x\ge 2 \end{cases}$

(i) WKT $\int_{-\infty}^{\infty} f(x)\,dx = 1 \Rightarrow \int_0^1 x\,dx + \int_1^2 k(4-2x)\,dx = 1$

$$\Rightarrow \left(\frac{x^2}{2}\right)_0^1 + k\left(4x-x^2\right)_1^2 = 1 \Rightarrow \frac{1}{2} + k(8-4-4+1) = 1$$

$$\Rightarrow \frac{1}{2} + k = 1 \Rightarrow k = 1/2$$

(ii) $P(A) = P[500 < X < 1500] = \int_{0.5}^{1.5} f(x)\,dx = \int_{0.5}^{1} x\,dx + \int_1^{1.5} \frac{1}{2}(4-2x)\,dx$

$$= \left(\frac{x^2}{2}\right)_{0.5}^{1} + \frac{1}{2}\left(4x-x^2\right)_1^{1.5} = 0.5 - 0.125 + \frac{1}{2}(3.75-3) = 0.75$$

$$P(B) = P[X>1000] = \int_1^2 \frac{1}{2}(4-2x)\,dx = \frac{1}{2}\left(4x-x^2\right)_1^2 = \frac{1}{2}(8-4-4+1)$$

$$= 1/2$$

(iii) $P(A\cap B) = P[1000 < X < 1500] = \int_{1}^{1.5} f(x)\,dx$

$$= \int_{1}^{1.5} \frac{1}{2}(4-2x)\,dx = \frac{1}{2}\left(4x - x^2\right)_{1}^{1.5} = \frac{1}{2}(3.75 - 4 + 1) = 0.375$$

$P(A).P(B) = (0.75)\left(\frac{1}{2}\right) = 0.375$

$\therefore P(A\cap B) = P(A).P(B)$

Hence A & B are independent.

⑩ Experience has shown that while walking in a certain park, the time X (in mins.), between seeing two people smoking has a density fun. of the form $f(x) = \begin{cases} \lambda x e^{-x}, & x > 0 \\ 0 & \text{elsewhere} \end{cases}$ (i) Calculate the value of λ. (ii) Find the distribution fun. of X.
(iii) What is the probability that Jeff, who has just seen a person smoking, will see another person smoking in 2 to 5 minutes? In atleast 7 minutes?

Sol: Given $f(x) = \begin{cases} \lambda x e^{-x}, & x > 0 \\ 0, & \text{otherwise} \end{cases}$

(i) WKT $\int_{-\infty}^{\infty} f(x)\,dx = 1 \Rightarrow \int_{0}^{\infty} \lambda x e^{-x}\,dx = 1 \Rightarrow \lambda\left[x\frac{e^{-x}}{-1} - e^{-x}\right]_{0}^{\infty} = 1$

$\Rightarrow \lambda(1) = 1 \Rightarrow \lambda = 1$

(ii) $F(x) = \int_{-\infty}^{x} f(x)\,dx = \int_{0}^{x} x e^{-x}\,dx = \left[x\frac{e^{-x}}{-1} - e^{-x}\right]_{0}^{x}$

$= -xe^{-x} - e^{-x} + 1 = 1 - e^{-x}(x+1),\ x > 0$

(iii) $P(2 < X < 5) = F(5) - F(2) = 1 - e^{-5}(6) - 1 + e^{-2}(3) = 0.3656$

$P(X \geq 7) = 1 - P(X < 7) = 1 - F(7) = 1 - 1 + e^{-7}(8) = 0.0073$

Moments - Moment Generating Functions & their Properties:

Moments (Discrete case):

Let X be a discrete R.V. taking the values $x_1, x_2, \ldots, x_n$ with probability mass fun. $p_1, p_2, \ldots, p_n$ respectively then the r^{th} moment about the origin is

μ_r' (about the origin) $= \sum_{i=1}^{n} x_i^r p_i$ —①

& μ_r' (about any point $x = A$) $= \sum_{i=1}^{n} (x_i - A)^r p_i$ —②

& μ_r (about mean) $= \sum_{i=1}^{n} (x_i - \text{mean})^r p_i$ —③

In particular from ①

$\mu_1' = \sum_{i=1}^{n} x_i p_i = \text{Mean } (\bar{x})$

$\mu_2' = \sum_{i=1}^{n} x_i^2 p_i = \text{Mean square value}$

Limitations of m.g.f:

① A r.v X may have no moments although its m.g.f. exists.

② A r.v. X can have m.g.f. & some or all moments; yet the m.g.f. does not generate the moments.

③ A r.v. X can have all or some moments, but m.g.f. does not exist except perhaps at one pt..

From ③, $\mu_2 = \sum_{i=1}^{n} (x_i - \text{mean})^2 p_i$ = Variance

$= \mu_2' - (\mu_1')^2 \quad (\because \bar{x} = \mu_1')$

$\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2\mu_1'^3$

$\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'\mu_1'^2 - 3\mu_1'^4$

Moments (Continuous case):

If X is a continuous R.V. with p.d.f. f(x) then defined in the interval (a,b). $\mu_r' = \int_a^b x^r f(x)dx$

μ_r' (about A) $= \int_a^b (x-A)^r f(x)dx$

μ_r (about mean) $= \int_a^b (x-\bar{x})^r f(x)dx$

Moment generating funḷ.: (m.g.f.).

The m.g.f. of a R.V. X (about origin) whose probability funḷ. f(x) is given by

$$M_X(t) = E[e^{tX}] = \begin{cases} \int_{-\infty}^{\infty} e^{tx} f(x)dx, & \text{for a continuous probability distribution} \\ \sum_{x}^{\infty} e^{tx} P(x), & \text{for a discrete probability distribution} \end{cases}$$

where t is real parameter...

To find the r^{th} moment of X about origin, we know that

$$M_X(t) = E[e^{tX}] = E\left[1 + \frac{tX}{1!} + \frac{(tX)^2}{2!} + \frac{(tX)^3}{3!} + \dots + \frac{(tX)^r}{r!} + \dots\right]$$

$$= 1 + E\left[\frac{tX}{1!}\right] + E\left[\frac{t^2X^2}{2!}\right] + \dots + E\left[\frac{t^rX^r}{r!}\right] + \dots$$

$$= 1 + tE(X) + \frac{t^2}{2!}E(X^2) + \dots + \frac{t^r}{r!}E(X^r) + \dots$$

(i) $M_X(t) = 1 + t\mu_1' + \frac{t^2}{2!}\mu_2' + \frac{t^3}{3!}\mu_3' + \dots + \frac{t^r}{r!}\mu_r' + \dots$ — ①

(ii) $M_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!}\mu_r'$. (using $\mu_r' = E(X^r)$)

This gives the m.g.f. interms of moments. Thus the coeffḷ. of $\frac{t^r}{r!}$ in $M_X(t)$ gives the r^{th} moment of the r.v. X about origin (μ_r'). Since $M_X(t)$ generates moments, it is known as moment generating funḷ..

Note: Diffḷ. ① w.r.t. t, we get

$M_X'(t) = \mu_1' + \frac{2t}{2!}\mu_2' + \frac{3t^2}{3!}\mu_3' + \dots$ — ②

Put t=0 in ②, we get $\boxed{\mu_1' = M_X'(0)}$

The first moment about origin is given by $\boxed{\mu_1' = M_x'(0) = \bar{x}}$, namely the mean.

Diff. ② w.r.t. t, we get

$$M_x''(t) = \mu_2' + t\mu_3' + \dots \quad \text{—③}$$

Put t=0 in ③, we get

$\boxed{M_x''(0) = \mu_2'}$. Hence the second moment about origin is given by $\boxed{\mu_2' = M_x''(0)}$

In general, we get $\mu_r' = \left[\frac{d^r}{dt^r}(M_x(t))\right]_{t=0}$

Note: The m.g.f. of X about the pt X=a is defined by

$$M_X(t) = E\left[e^{t(X-a)}\right] = E\left[1+t(X-a)+\frac{t^2}{2!}(X-a)^2+\dots+\frac{t^r}{r!}(X-a)^r+\dots\right]$$

$$= 1+E\left[t(X-a)\right]+E\left[\frac{t^2}{2!}(X-a)^2\right]+\dots+E\left[\frac{t^r}{r!}(X-a)^r\right]+\dots$$

$$= 1+tE(X-a)+\frac{t^2}{2!}E(X-a)^2+\dots+\frac{t^r}{r!}E(X-a)^r+\dots$$

$$= 1+t\mu_1'+\frac{t^2}{2!}\mu_2'+\dots+\frac{t^r}{r!}\mu_r'+\dots$$

Thus $[M_X(t)]_{X=a} = 1+t\mu_1'+\frac{t^2}{2!}\mu_2'+\dots+\frac{t^r}{r!}\mu_r'+\dots$ where $\mu_r' = E\left[(X-a)^r\right]$

which gives the rth moment about the pt x=a.

Properties of m.g.f.:

① Let Y=aX+b, where X is a R.V. with m.g.f. $M_X(t)$. Then

$$M_Y(t) = E\left[e^{tY}\right] = E\left[e^{t(aX+b)}\right] = E\left[e^{tax}.e^{bt}\right]$$

$$= e^{bt}E\left[e^{Xat}\right] = e^{bt}M_X(at)$$

② $M_{cX}(t) = E\left[e^{cXt}\right] = E\left[e^{X(ct)}\right] = M_X(ct)$ where c is a constant.

③ If X & Y are two independent random variables, then $M_{X+Y}(t) = M_X(t).M_Y(t)$.

Proof: $M_{X+Y}(t) = E\left[e^{t(X+Y)}\right] = E\left[e^{tX+tY}\right] = E\left[e^{tX}.e^{tY}\right]$

$= E\left[e^{tX}\right].E\left[e^{tY}\right]$ (∵ X & Y are independent)

$= M_X(t).M_Y(t)$

④ If $X_1, X_2, \dots, X_n$ are n independent RVs then

$$M_{X_1+X_2+\dots+X_n}(t) = M_{X_1}(t)\,M_{X_2}(t)\dots M_{X_n}(t)$$

Sol: $M_{X_1+X_2+\dots+X_n}(t) = E\left[e^{(X_1+X_2+\dots+X_n)t}\right] = E\left[e^{X_1t}e^{X_2t}\dots e^{X_nt}\right]$

$= E\left[e^{X_1t}\right]E\left[e^{X_2t}\right]\dots E\left[e^{X_nt}\right]$ (∵ X_i's are independent)

$= M_{X_1}(t)\,M_{X_2}(t)\dots M_{X_n}(t)$

Problems:

① For the triangular distribution $f(x) = \begin{cases} x, & 0<x\le 1 \\ 2-x, & 1\le x<2 \\ 0, & \text{otherwise} \end{cases}$ find the mean, variance & the m.g.f.

Sol: Given $f(x) = \begin{cases} x, & 0<x\le 1 \\ 2-x, & 1\le x<2 \\ 0, & \text{otherwise} \end{cases}$

$$\text{Mean} = E(X) = \int_{-\infty}^{\infty} x f(x)\,dx = \int_0^1 x^2\,dx + \int_1^2 x(2-x)\,dx$$

$$= \left(\frac{x^3}{3}\right)_0^1 + \left(x^2 - \frac{x^3}{3}\right)_1^2 = \frac{1}{3} + 4 - \frac{8}{3} - 1 + \frac{1}{3} = 1$$

$$E[X^2] = \int_{-\infty}^{\infty} x^2 f(x)\,dx = \int_0^1 x^3\,dx + \int_1^2 x^2(2-x)\,dx = \int_0^1 x^3\,dx + \int_1^2 (2x^2 - x^3)\,dx$$

$$= \left(\frac{x^4}{4}\right)_0^1 + \left(\frac{2x^3}{3} - \frac{x^4}{4}\right)_1^2 = \frac{1}{4} + \frac{16}{3} - 4 - \frac{2}{3} + \frac{1}{4} = \frac{1}{2} + \frac{14}{3} - 4$$

$$= \frac{3+28-24}{6} = \frac{7}{6}$$

$$\text{Var}(X) = E(X^2) - [E(X)]^2 = \frac{7}{6} - 1 = \frac{1}{6}$$

The m.g.f. of the r.v. X is

$$M_X(t) = E[e^{tX}] = \int_{-\infty}^{\infty} e^{tx} f(x)\,dx = \int_0^1 x e^{tx}\,dx + \int_1^2 (2-x) e^{tx}\,dx$$

$$= \left[x\frac{e^{tx}}{t} - \frac{e^{tx}}{t^2}\right]_0^1 + \left[(2-x)\frac{e^{tx}}{t} - (-1)\frac{e^{tx}}{t^2}\right]_1^2$$

$$= \frac{e^t}{t} - \frac{e^t}{t^2} + \frac{1}{t^2} + \frac{e^{2t}}{t^2} - \frac{e^t}{t} - \frac{e^t}{t^2} = \frac{e^{2t}}{t^2} - \frac{2e^t}{t^2} + \frac{1}{t^2}$$

$$= \frac{1}{t^2}\left[e^{2t} - 2e^t + 1\right] = \frac{1}{t^2}\left[e^t - 1\right]^2$$

② Let X be a RV with probability law $P(X=r) = q^{r-1}p$; $r = 1, 2, 3, \ldots$. Find the m.g.f. & hence mean & variance assume $p+q=1$.

Sol: WKT $M_X(t) = E[e^{tX}] = \sum_{x=1}^{\infty} e^{tx} P(x) = \sum_{r=1}^{\infty} e^{tr} P(r)$

$$= \sum_{r=1}^{\infty} e^{tr} q^{r-1} p = \frac{p}{q} \sum_{r=1}^{\infty} e^{tr} q^r = \frac{p}{q} \sum_{r=1}^{\infty} (qe^t)^r$$

$$= \frac{p}{q}\left[qe^t + (qe^t)^2 + (qe^t)^3 + \ldots\right] = \frac{p}{q} qe^t \left[1 + qe^t + (qe^t)^2 + \ldots\right]$$

$$= pe^t\left[1 - qe^t\right]^{-1}$$

$$\therefore M_X(t) = \frac{pe^t}{1 - qe^t}$$

$$M_X'(t) = \frac{(1-qe^t)pe^t - pe^t(-qe^t)}{(1-qe^t)^2}$$

$$\text{Mean} = M_X'(0) = \frac{(1-q)p - p(-q)}{(1-q)^2} = \frac{p - pq + pq}{(1-q)^2} = \frac{p}{p^2} = \frac{1}{p} \quad (\because p+q=1)$$

$$M_x'(t) = \frac{pe^t - pqe^{2t} + pqe^{2t}}{(1-qe^t)^2} = \frac{pe^t}{(1-qe^t)^2}$$

$$M_x''(t) = \frac{(1-qe^t)^2 pe^t - pe^t . 2(1-qe^t)(-qe^t)}{(1-qe^t)^4} = \frac{pe^t - pqe^{2t} + 2pqe^{2t}}{(1-qe^t)^3} = \frac{pe^t + pqe^{2t}}{(1-qe^t)^3}$$

$$M_x''(0) = \frac{p+pq}{(1-q)^3} = \frac{p(1+q)}{p^3} = \frac{1+q}{p^2}$$

$$Var(X) = M_x''(0) - [M_x'(0)]^2 = \frac{1+q}{p^2} - \frac{1}{p^2} = \frac{q}{p^2}$$

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③ The first 4 moments of a distribution about $X=4$ are 1, 4, 10 & 45 respectively. S.T. the mean is 5, variance is 3, $\mu_3 = 0$ & $\mu_4 = 26$.

Sol: Given that $\mu_1' = 1$, $\mu_2' = 4$, $\mu_3' = 10$ & $\mu_4' = 45$; $A = 4$

$$\mu_1 = \text{Mean} = A + \mu_1' = 4 + 1 = 5$$

$$\text{Variance} = \mu_2 = \mu_2' - \mu_1'^2 = 4 - \cancel{25}\,1 = 3$$

$$\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2\mu_1'^3 = 10 - 3(4)(1) + 2(1)^3 = 10 - 12 + 2 = 0$$

$$\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'\mu_1'^2 - 3\mu_1'^4 = 45 - 4(10)(1) + 6(4)(1)^2 - 3(1)^4$$

$$= 45 - 40 + 24 - 3 = 26$$

---

④ Find the m.g.f. of an exponential r.v. & hence find its mean & variance.

Sol: The p.d.f. of an exponential distribution is given by

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

The m.g.f. is given by

$$M_X(t) = E[e^{tX}] = \int_{-\infty}^{\infty} e^{tx} f(x)\,dx = \int_{0}^{\infty} e^{tx} \lambda e^{-\lambda x}\,dx$$

$$= \lambda \int_0^\infty e^{tx} e^{-\lambda x}\,dx = \lambda \int_0^\infty e^{(t-\lambda)x}\,dx = \lambda \left[\frac{e^{(t-\lambda)x}}{t-\lambda} \right]_0^\infty$$

$$= \lambda \left[\frac{e^{-(\lambda - t)x}}{t-\lambda} \right]_0^\infty = \frac{\lambda}{t-\lambda}[0-1] = \frac{\lambda}{\lambda - t}$$

Mean $= M_x'(0)$.

$$M_x'(t) = \frac{(\lambda - t).0 - \lambda(-1)}{(\lambda - t)^2} = \frac{\lambda}{(\lambda - t)^2}$$

$$\therefore \text{Mean} = M_x'(0) = \frac{\lambda}{\lambda^2} = \frac{1}{\lambda}$$

$$M_x''(t) = \frac{(\lambda - t)^2 . 0 - \lambda 2(\lambda - t).(-1)}{(\lambda - t)^4} = \frac{2\lambda(\lambda - t)}{(\lambda - t)^4} = \frac{2\lambda}{(\lambda - t)^3}$$

The second moment $= M_x''(0) = \frac{2\lambda}{\lambda^3} = \frac{2}{\lambda^2}$

Variance $= M_x''(0) - [M_x'(0)]^2 = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}$

---

⑤ The density fun. of a r.v. X is given by $f(x) = kx(2-x),\ 0 \le x \le 2$. Find k, mean, variance & r^{th} moment.

Sol: Given $f(x) = kx(2-x),\ 0 \le x \le 2$ is a p.d.f.

WKT $\int_{-\infty}^{\infty} f(x)dx = 1 \Rightarrow k\int_0^2 x(2-x)dx = 1 \Rightarrow k\int_0^2 (2x - x^2)dx = 1$

$$\Rightarrow k\left(x^2 - \frac{x^3}{3}\right)_0^2 = 1 \Rightarrow k\left(4 - \frac{8}{3}\right) = 1 \Rightarrow k = 3/4$$

$$\text{Mean} = E(X) = \int_{-\infty}^{\infty} x f(x)dx = \frac{3}{4}\int_0^2 x^2(2-x)dx = \frac{3}{4}\int_0^2 (2x^2 - x^3)dx$$

$$= \frac{3}{4}\left[\frac{2x^3}{3} - \frac{x^4}{4}\right]_0^2 = \frac{3}{4}\left[\frac{16}{3} - 4\right] = 1$$

$$E[X^2] = \int_{-\infty}^{\infty} x^2 f(x)dx = \frac{3}{4}\int_0^2 x^3(2-x)dx = \frac{3}{4}\int_0^2 (2x^3 - x^4)dx$$

$$= \frac{3}{4}\left[\frac{x^4}{2} - \frac{x^5}{5}\right]_0^2 = \frac{3}{4}\left[8 - \frac{32}{5}\right] = \frac{3}{4} \times \frac{8}{5} = \frac{6}{5}$$

$$Var(X) = E(X^2) - [E(X)]^2 = \frac{6}{5} - 1 = \frac{1}{5}$$

$$\mu_r' = E[X^r] = \int_{-\infty}^{\infty} x^r f(x)dx = \frac{3}{4}\int_0^2 x^r x(2-x)dx = \frac{3}{4}\int_0^2 (2x^{r+1} - x^{r+2})dx$$

$$= \frac{3}{4}\left[2\frac{x^{r+2}}{r+2} - \frac{x^{r+3}}{r+3}\right]_0^2 = \frac{3}{4}\left[2\frac{(2)^{r+2}}{r+2} - \frac{(2)^{r+3}}{r+3}\right]$$

$$= \frac{3}{4}(2)^{r+3}\left[\frac{1}{r+2} - \frac{1}{r+3}\right] = 6(2)^r\left[\frac{r+3-r-2}{(r+2)(r+3)}\right]$$

$$= \frac{6\,2^r}{(r+2)(r+3)}$$

---

⑥ A continuous r.v. X has the p.d.f. f(x) given by $f(x) = ce^{-|x|}$, ~~$-\alpha_1 < x < \alpha_1$~~. $-\infty < x < \infty$
Find the value of c & m.g.f.

Sol: Given that $f(x) = ce^{-|x|}$ $-\infty < x < \infty$ ~~$-\alpha_1 < x < \alpha_1$~~ is a p.d.f.

WKT $\int_{-\infty}^{\infty} f(x)dx = 1 \Rightarrow \int_{-\infty}^{\infty} ce^{-|x|}dx = 1 \Rightarrow 2c\int_0^{\infty} e^{-x}dx = 1$

$$\Rightarrow 2c\left(\frac{e^{-x}}{-1}\right)_0^{\infty} = 1 \Rightarrow -2c(0-1) = 1 \Rightarrow c = 1/2$$

$$\therefore f(x) = \frac{1}{2}e^{-|x|}$$

$$M_X(t) = E[e^{tX}] = \int_{-\infty}^{\infty} e^{tx} f(x)\,dx = \int_{-\infty}^{\infty} e^{tx}\cdot\frac{1}{2}e^{-|x|}\,dx = \frac{1}{2}\int_{-\infty}^{\infty} e^{tx} e^{-|x|}\,dx$$

$$= \frac{1}{2}\left[\int_{-\infty}^{0} e^{tx} e^{x}\,dx + \int_{0}^{\infty} e^{tx} e^{-x}\,dx\right] = \frac{1}{2}\left[\int_{-\infty}^{0} e^{(t+1)x}\,dx + \int_{0}^{\infty} e^{(t-1)x}\,dx\right]$$

$$= \frac{1}{2}\left[\left(\frac{e^{(t+1)x}}{t+1}\right)_{-\infty}^{0} + \left(\frac{e^{(t-1)x}}{t-1}\right)_{0}^{\infty}\right]$$

$$= \frac{1}{2}\left[\left(\frac{e^{(t+1)x}}{t+1}\right)_{-\infty}^{0} + \left(\frac{e^{-(1-t)x}}{t-1}\right)_{0}^{\infty}\right]$$

$$= \frac{1}{2}\left[\frac{1}{t+1}(1-0) + \frac{1}{t-1}(0-1)\right] = \frac{1}{2}\left[\frac{1}{t+1} - \frac{1}{t-1}\right]$$

$$= \frac{1}{2}\left[\frac{t-1-t-1}{(t+1)(t-1)}\right] = \frac{-1}{t^2-1} = \frac{1}{1-t^2}$$

Binomial Distribution:

Bernoulli Trial:

Each trial has two possible outcomes, generally called success & failure. Such a trial is known as Bernoulli trial. The sample space for a Bernoulli trial is $S = \{s, f\}$.

E.g: (i) A toss of a single coin (head or tail)
(ii) The throw of a die (even or odd no/.)

Binomial experiment:

An experiment consisting of a repeated no/. of Bernoulli trials is called Binomial experiment. A binomial experiment must possess the following properties. (i) There must be a fixed no/. of trials.
(ii) All trials must have identical probabilities of success (p).
(iii) The trials must be independent of each other.

Binomial distribution:

Consider a set of n independent Bernoullian trials (n being finite), in which the probability p of success in any trial is constant for each trial. Then $q = 1-p$ is the probability of failure in any trial. A r.v. X is said to follow binomial distribution if it assumes only non-(-)ve values & its probability mass fun/. is given by

$$P(X=x) = p(x) = \begin{cases} {}^nC_x\, p^x q^{n-x} & , x = 0, 1, 2, \ldots, n, \ q = 1-p \\ 0 & , \text{otherwise} \end{cases}$$

The 2 independent constants n & p in the distribution are known as the parameters of the distribution. 'n' is also, sometimes known as the degree of the

(binomial distribution.

Binomial distribution: $P(X=x) = p(x) = nC_x\, p^x q^{n-x}$

The m.g.f. $M_X(t) = E[e^{tX}] = \sum_{x=0}^{n} e^{tx}\, nC_x\, p^x q^{n-x}$

$$= \sum_{x=0}^{n} nC_x (pe^t)^x q^{n-x}$$

$$= nC_0 (pe^t)^0 q^n + nC_1 (pe^t) q^{n-1} + nC_2 (pe^t)^2 q^{n-2} + \dots + nC_n (pe^t)^n q^0$$

$$= q^n + nC_1 (pe^t) q^{n-1} + nC_2 (pe^t)^2 q^{n-2} + \dots + (pe^t)^n$$

$$= (pe^t + q)^n$$

Mean $= E(X) = \left[\frac{d}{dt}(M_X(t))\right]_{t=0} = \left[\frac{d}{dt}(pe^t+q)^n\right]_{t=0}$

$$= \left[n(pe^t+q)^{n-1} pe^t\right]_{t=0} = n(p+q)^{n-1} p = np \quad (\because p+q=1)$$

$E(X^2) = \left[\frac{d^2}{dt^2}(M_X(t))\right]_{t=0} = \left[\frac{d}{dt}\left(np(pe^t+q)^{n-1}e^t\right)\right]_{t=0}$

$$= np\left[(pe^t+q)^{n-1}e^t + e^t(n-1)(pe^t+q)^{n-2}pe^t\right]_{t=0}$$

$$= np\left[(p+q)^{n-1} + (n-1)(p+q)^{n-2}p\right] = np\left[1+(n-1)p\right] = np + np^2(n-1)$$

$$= np + n^2p^2 - np^2$$

$Var(X) = E(X^2) - [E(X)]^2 = np + n^2p^2 - np^2 - n^2p^2 = np - np^2 = np(1-p) = npq$

Problems:

① The mean & variance of a binomial variate are 8 & 6. Find $P(X \geq 2)$.

Sol: Given Mean $= np = 8$; Variance $= npq = 6$

$$\frac{npq}{np} = \frac{6}{8} \Rightarrow q = 3/4$$

$$p = 1 - q = 1 - \frac{3}{4} = \frac{1}{4}$$

$$np = 8 \Rightarrow n/4 = 8 \Rightarrow n = 32$$

$$P(x) = nC_x\, p^x q^{n-x} = 32C_x \left(\frac{1}{4}\right)^x \left(\frac{3}{4}\right)^{32-x}$$

$P(X \geq 2) = 1 - P(X<2) = 1 - [P(X=0) + P(X=1)]$

$$= 1 - \left[32C_0 \left(\frac{1}{4}\right)^0 \left(\frac{3}{4}\right)^{32} + 32C_1 \left(\frac{1}{4}\right)^1 \left(\frac{3}{4}\right)^{31}\right]$$

$$= 1 - \left[\left(\frac{3}{4}\right)^{32} + 32\left(\frac{1}{4}\right)\left(\frac{3}{4}\right)^{31}\right] = 1 - \left(\frac{3}{4}\right)^{31}\left(\frac{3}{4} + \frac{32}{4}\right)$$

$$= 1 - \frac{35}{4}\left(\frac{3}{4}\right)^{31}$$

---

② Find the probability that in tossing a fair coin 5 times, there will appear (a) 3 heads (b) 3 tails & 2 heads (c) atleast 1 head & (d) not more than 1 tail.

Sol: $p = \frac{1}{2}$, $q = \frac{1}{2}$ & $n = 5$

WKT $P(X=x) = nC_x p^x q^{n-x}$

(a) $P(\text{getting 3 heads}) = P(X=3) = 5C_3 \left(\frac{1}{2}\right)^3 \left(\frac{1}{2}\right)^2 = \frac{5\times4\times3}{1\times2\times3} \cdot \frac{1}{2^5} = \frac{5}{16}$

(b) We note that getting 3 tails & 2 heads is equivalent to getting 3 tails or 2 heads.

$P(\text{getting 3 tails \& 2 heads}) = P(\text{getting 2 heads})$

$= P(X=2) = 5C_2 \left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)^3 = 10 \cdot \frac{1}{2^5} = \frac{5}{16}$

(c) $P(\text{getting atleast 1 head}) = P(X \geq 1) = 1 - P(X<1) = 1 - P(X=0)$

$= 1 - 5C_0 \left(\frac{1}{2}\right)^0 \left(\frac{1}{2}\right)^5 = 1 - \frac{1}{32} = \frac{31}{32}$.

(d) $P(\text{not more than 1 tail}) = P(\text{getting 0 tail}) + P(\text{getting 1 tail})$

$= P(\text{getting all heads}) + P(\text{getting 4 heads})$

$= 5C_5 \left(\frac{1}{2}\right)^5 \left(\frac{1}{2}\right)^0 + 5C_4 \left(\frac{1}{2}\right)^4 \left(\frac{1}{2}\right)^1$

$= \frac{1}{2^5} + 5 \cdot \frac{1}{2^5} = \frac{6}{2^5} = \frac{3}{16}$

---

③ An irregular 6 faced die is thrown such that the probability that it gives 3 even nos/. in 5 throws is twice the probability that it gives 2 even nos/. in 5 throws. How many sets of exactly 5 trials can be expected to give no even no/. out of 2500 sets.

Sol: WKT $P(X = x \text{ successes}) = nC_x p^x q^{n-x}$

Here p - the probability of getting an even no/. in a throw of a die.

Given P(getting 3 even nos/. in 5 throws) = 2P(getting 2 even nos/. in 5 throws)

(ie) $P(X=3) = 2P(X=2)$

$5C_3 p^3 q^2 = 2 \times 5C_2 p^2 q^3$ (Here $n=5$)

$\Rightarrow 10p^3q^2 = 20p^2q^3 \Rightarrow p = 2q \Rightarrow p = 2(1-p) = 2-2p$

$\Rightarrow 3p = 2 \Rightarrow p = \frac{2}{3}$

$q = 1 - p = 1 - \frac{2}{3} = \frac{1}{3}$

$\therefore P(\text{getting no even no/.}) = P(X=0) = 5C_0 \left(\frac{2}{3}\right)^0 \left(\frac{1}{3}\right)^5 = \frac{1}{3^5}$

In 2500 sets the no/. of sets having no even no/. is $= 2500 \times P(X=0)$

$= 2500 \times \frac{1}{3^5} = 10.2881$

④ If 10% of the screws produced by an automatic machine are defective, find the probability that out of 20 screws selected at random, there are (i) exactly 2 defective (ii) atmost 3 defective (iii) atleast 2 defectives & (iv) between 1 & 3 defectives (inclusive).

Sol: Given P(a screw is defective) $= \frac{10}{100}$ (ie) $p = \frac{1}{10}$

$$\therefore q = 1-p = 1-\frac{1}{10} = \frac{9}{10}$$

$$n=20$$

WKT $P(X=x) = nC_x\, p^x q^{n-x}$

(i) P(exactly 2 defectives) $= P(X=2) = 20C_2 \left(\frac{1}{10}\right)^2 \left(\frac{9}{10}\right)^{18} = 190 \times \frac{9^{18}}{10^{20}}$

(ii) P(atmost 3 defective) $= P(X\leq 3) = P(X=0)+P(X=1)+P(X=2)+P(X=3)$

$$= 20C_0 \left(\frac{1}{10}\right)^0 \left(\frac{9}{10}\right)^{20} + 20C_1 \left(\frac{1}{10}\right)^1 \left(\frac{9}{10}\right)^{19} + 20C_2 \left(\frac{1}{10}\right)^2 \left(\frac{9}{10}\right)^{18} + 20C_3 \left(\frac{1}{10}\right)^3 \left(\frac{9}{10}\right)^{17}$$

$$= \left(\frac{9}{10}\right)^{20} + 20\left(\frac{1}{10}\right)\left(\frac{9}{10}\right)^{19} + 190\left(\frac{1}{10}\right)^2\left(\frac{9}{10}\right)^{18} + 1140\left(\frac{1}{10}\right)^3\left(\frac{9}{10}\right)^{17}$$

$$= \frac{9^{17}}{10^{20}}\left(9^3+(20\times 9^2)+(190\times 9)+(1140\times 1)\right) = 5199 \times \frac{9^{17}}{10^{20}}$$

(iii) P(atleast 2 defectives) $= P(X\geq 2) = 1-P(X<2) = 1-[P(X=0)+P(X=1)]$

$$= 1-\left[\left(\frac{9}{10}\right)^{20} + 20\left(\frac{1}{10}\right)\left(\frac{9}{10}\right)^{19}\right] = 1-\frac{9^{19}}{10^{20}}(9+20) = 1-\frac{9^{19}}{10^{20}}\cdot 29$$

(iv) P(no. of defectives lies between 1 & 3 (inclu)) $= P(1\leq X\leq 3)$

$$= P(X=1)+P(X=2)+P(X=3)$$

$$= 20C_1\left(\frac{1}{10}\right)^1\left(\frac{9}{10}\right)^{19} + 20C_2\left(\frac{1}{10}\right)^2\left(\frac{9}{10}\right)^{18} + 20C_3\left(\frac{1}{10}\right)^3\left(\frac{9}{10}\right)^{17}$$

$$= 20\times\frac{9^{19}}{10^{20}} + 190\times\frac{9^{18}}{10^{20}} + 1140\times\frac{9^{17}}{10^{20}}$$

$$= \frac{9^{17}}{10^{20}}\left[1620+1710+1140\right] = \frac{9^{17}}{10^{20}}(4470)$$

⑤ In a certain town, 20% samples of the population is literate & assume that 200 investigators take samples of ten individuals to see whether they are literate. How many investigators would you expect to report that 3 people or less literates in the samples?

Sol: Given P(an individual is literate) $= \frac{20}{100} = 0.2$

(ie) $p=0.2$

$q = 1-p = 1-0.2 = 0.8$ & $n=10$ (sample size)

WKT $P(X=x) = nC_x\, p^x q^{n-x}$

P(an investigator reporting 3 or less as literate) $= P(X\leq 3)$

$$= P(X=0)+P(X=1)+P(X=2)+P(X=3)$$

$= 10C_0(0.2)^0(0.8)^{10} + 10C_1(0.2)^1(0.8)^9 + 10C_2(0.2)^2(0.8)^8 + 10C_3(0.2)^3(0.8)^7$

$= (0.8)^7\left[0.512 + (10\times0.2\times0.64) + (45\times0.04\times0.8) + (120\times0.008\times1)\right]$

$= (0.8)^7(4.192) = 0.8791$

∴ (200 investigator reporting 3 or less as literate) = $200\times0.8791 = 175.82$

---

⑥ It is know that screws produced by a certain company will be defective with probability 0.01 independently of each other. The company sells the screws in packages of 10 & offers a money-back guarantee that atmost 1 of the 10 screws is defective. What proportion of packages sold must the company replace?

Sol: Given $p = 0.01$; $q = 1 - p = 1 - 0.01 = 0.09$; $n = 10$

WKT $P(X=x) = nC_x\, p^x q^{n-x}$

∴ P(atmost 1 screw is defective) $= P(X\le1) = P(X=0) + P(X=1)$

$= 10C_0(0.01)^0(0.09)^{10} + 10C_1(0.01)^1(0.09)^9$

$= (0.09)^9(0.09 + (10\times0.01)) = (0.09)^9(0.19)$

∴ P(a package will have to replace) $= 1 - P(X\le1) = 1 - (0.09)^9(0.19) = 1$

∴ 1% of the packages will have to replace.

---

⑦ Suppose that the r.v. X is equal to the no. of hits obtained by a certain base ball player in his next 3 bats. If $P(X=1) = 0.3$, $P(X=2) = 0.2$ & $P(X=0) = 3P(X=3)$ Find $E(X)$.

Sol: WKT $P(X=0) + P(X=1) + P(X=2) + P(X=3) = 1$ — ①

Given $P(X=0) = 3P(X=3)$ — ②

Subst. ② in ① we get $3P(X=3) + 0.3 + 0.2 + P(X=3) = 1$

$\Rightarrow 4P(X=3) = 0.5 \Rightarrow P(X=3) = 0.125$

Given $P(X=0) = 3P(X=3) = 3\times0.125 = 0.375$

WKT $E(X) = \sum_i x_i P(x_i) = (1\times P(X=1)) + 2P(X=2) + 3P(X=3)$

$= 1(0.3) + 2(0.2) + 3(0.125) = 1.075$

---

⑧ 6 dice are thrown 729 times. How many times do you expect atleast three dice to show a five or a six?

Sol: p = probability of getting 5 or 6 with one die $= \frac{2}{6} = \frac{1}{3}$

$\therefore q = 1 - p = 1 - \frac{1}{3} = \frac{2}{3}$

P(atleast 3 dice showing 5 or 6) $= P(X\ge3) = P(X=3) + P(X=4) + P(X=5)$
$+ P(X=6)$

$$= 6C_3\left(\frac{1}{3}\right)^3\left(\frac{2}{3}\right)^3 + 6C_4\left(\frac{1}{3}\right)^4\left(\frac{2}{3}\right)^2 + 6C_5\left(\frac{1}{3}\right)^5\left(\frac{2}{3}\right)^1 + 6C_6\left(\frac{1}{3}\right)^6\left(\frac{2}{3}\right)^0$$

$$=(20\times 8)\frac{1}{3^6}+(15\times 4)\frac{1}{3^6}+(6\times 2)\frac{1}{3^6}+\frac{1}{3^6}$$

$$=\frac{1}{3^6}(160+60+12+1) = \frac{233}{3^6}$$

For 729 times, the expected no. of times atleast 3 dice showing five or six $= N\times\frac{233}{3^6} = 729\times\frac{233}{3^6} = 233$ times

---

⑨ The probability of a bomb hitting a target is $\frac{1}{5}$. Two bombs are enough to destroy a bridge. If six bombs are aimed at the bridge, find the probability that the bridge is destroyed?

Sol: Given P(hitting the target) $= \frac{1}{5}$ (ie) $p=\frac{1}{5}$

$q = 1-p = 1-\frac{1}{5} = \frac{4}{5}$; $n=6$

WKT $P(X=x) = nC_x p^x q^{n-x}$

P(the bridge is destroyed) $= P(X=2) = 6C_2\left(\frac{1}{5}\right)^2\left(\frac{4}{5}\right)^4 = 15\times 4^4\times\frac{1}{5^6}$

$= 0.2458$

Poisson Distribution:

Poisson distribution is a limiting case of binomial distribution under the following assumptions.

(i) The no. of trails 'n' should be indefinitely large. (ie) $n\to\infty$

(ii) The probability of successes 'p' for each trail is indefinitely small.

(iii) $np=\lambda$, should be finite where λ is a constant.

Derivation: Poisson distribution is given by $P(X=x) = p(x) = \frac{e^{-\lambda}\lambda^x}{x!}$

The m.g.f. $M_x(t) = \sum_{x=0}^{\infty} e^{tx} p(x) = \sum_{x=0}^{\infty} e^{tx}\frac{e^{-\lambda}\lambda^x}{x!}$

$$= \sum_{x=0}^{\infty}\frac{(\lambda e^t)^x e^{-\lambda}}{x!} = e^{-\lambda}\sum_{x=0}^{\infty}\frac{(\lambda e^t)^x}{x!}$$

$$= e^{-\lambda}\left[1+\frac{\lambda e^t}{1!}+\frac{(\lambda e^t)^2}{2!}+\cdots\right]$$

$$= e^{-\lambda}e^{\lambda e^t} = e^{\lambda(e^t-1)}$$

Mean $= E(X) = \left[\frac{d}{dt}[M_x(t)]\right]_{t=0} = \left[\frac{d}{dt}\left(e^{\lambda(e^t-1)}\right)\right]_{t=0}$

$$=\left[\frac{d}{dt}\left[e^{\lambda e^t}.e^{-\lambda}\right]\right]_{t=0} = e^{-\lambda}\left[e^{\lambda e^t}.\lambda e^t\right]_{t=0}$$

$$= e^{-\lambda}\lambda e^{\lambda} = \lambda$$

$$E[X^2] = \left[\frac{d^2}{dt^2}M_x(t)\right]_{t=0} = e^{-\lambda}\left[\frac{d}{dt}\left[e^{\lambda e^t}.\lambda e^t\right]\right]_{t=0} = \lambda e^{-\lambda}\left[\frac{d}{dt}\left[e^{\lambda e^t}.e^t\right]\right]_{t=0}$$

$$= \lambda e^{-\lambda}\left[e^{\lambda e^t}.e^t + e^t.e^{\lambda e^t}\lambda e^t\right]_{t=0}$$

$$= \lambda e^{-\lambda}\left[e^{\lambda} + \lambda e^{\lambda}\right] = \lambda[1+\lambda] = \lambda + \lambda^2$$

$$Var(X) = E[X^2] - [E(X)]^2 = \lambda + \lambda^2 - \lambda^2 = \lambda$$

Problems:

① If X is a Poisson variate such that $P(X=1) = \frac{3}{10}$ & $P(X=2) = \frac{1}{5}$, find $P(X=0)$ & $P(X=3)$.

Sol: WKT $P(X=x) = \frac{e^{-\lambda}\lambda^x}{x!}$

$$P(X=1) = e^{-\lambda}\lambda = \frac{3}{10} \quad ; \quad P(X=2) = \frac{e^{-\lambda}\lambda^2}{2!} = \frac{1}{5} \Rightarrow e^{-\lambda}\lambda^2 = 2/5$$

$$\therefore \frac{e^{-\lambda}\lambda^2}{e^{-\lambda}\lambda} = \frac{2/5}{3/10} \Rightarrow \lambda = \frac{2}{5}\times\frac{10}{3} = 4/3$$

$$P(X=0) = \frac{e^{-\lambda}\lambda^0}{0!} = e^{-\lambda} = e^{-4/3}$$

$$P(X=3) = \frac{e^{-\lambda}\lambda^3}{3!} = \frac{e^{-4/3}(4/3)^3}{6} = e^{-4/3}\frac{4^3}{3^3\times 6} = e^{-4/3}\left(\frac{32}{81}\right)$$

---

② One-fifth percent of the blades produced by a blade manufacturing factory turn out to be defective. The blades are in packets of 10. Use poisson distribution to calculate the approximate no. of packets containing (i) no defective (ii) one defective (iii) 2 defective blades respectively in a consignment of 10,000 packets.

Sol: Given $p = \frac{1/5}{100} = \frac{1}{500}$, $n = 10$, $N = 10000$

$$Mean = \lambda = np = 10\times\frac{1}{500} = \frac{1}{50} = 0.02$$

The Poisson distribution is $P(x) = \frac{e^{-\lambda}\lambda^x}{x!} = \frac{e^{-0.02}(0.02)^x}{x!}$

(i) $P(\text{no defective}) = P(0) = \frac{e^{-0.02}(0.02)^0}{0!} = e^{-0.02} = 0.9802$

∴ The total no. of packets containing no defective blades in a consignment of 10000 packets = N×P(no defective) = 10000×0.9802 = 9802 packets

(ii) $P(\text{one defective}) = P(1) = \dfrac{e^{-0.02}(0.02)^1}{1!} = 0.0196$

$\therefore$ No/. of packets containing one defective = N×P(one defective)

$= 10000 \times 0.0196 = 196$ packets

(iii) $P(\text{two defective}) = P(2) = \dfrac{e^{-0.02}(0.02)^2}{2!} = 0.0002$

$\therefore$ No/. of packets containing 2 defectives = N×P(2 defectives)

$= 10000 \times 0.0002 = 2$ packets

③ Six coins are tossed 6400 times. Using the poisson distribution, what is the approximate probability of getting six heads 10 times.

Sol: Given $n = 6400$

Probability of getting one head with one coin $= \frac{1}{2}$

$\therefore$ The probability of getting six heads with six coins $= \left(\frac{1}{2}\right)^6 = \frac{1}{64}$

$\therefore$ Mean $= \lambda = np = 6400 \times \frac{1}{64} = 100$

The poisson distribution is $P(x) = \dfrac{e^{-\lambda}\lambda^x}{x!} = \dfrac{e^{-100}(100)^x}{x!}$

(ie) $P(\text{getting } x \text{ heads}) = \dfrac{e^{-100}(100)^x}{x!}$

$\therefore$ Probability of getting 6 heads 10 times $= P(X=10) = \dfrac{e^{-100}(100)^{10}}{10!}$

④ If the m.g.f. of the r.v. X is $e^{4(e^t-1)}$, find $P(X = \mu+\sigma)$ where μ & σ^2 are the mean & variance of the poisson distribution.

Sol: The m.g.f. of a poisson distribution fun/. $M_X(t) = e^{\mu(e^t-1)}$

where $\mu = \text{Mean} = 4$

Standard deviation. $\sigma = \sqrt{\text{Variance}} = \sqrt{\text{Mean}}$ [Mean = Variance for a poisson distribution]

$\sigma = \sqrt{4} = 2$

$\therefore P(X = \mu+\sigma) = P(X = 4+2) = P(X=6) = \dfrac{e^{-4}(4)^6}{6!} = 0.1042$

⑤ If X is a Poisson variate such that $P(X=2) = 9P(X=4) + 90P(X=6)$, find the variance.

Sol: The probability distribution for the poisson r.v. X is given by

$$P(X=x) = \frac{e^{-\lambda}\lambda^{x}}{x!}, \quad x = 0, 1, \dots \; \& \; \lambda > 0$$

Given that $P(X=2) = 9P(X=4) + 90P(X=6)$

$$\frac{e^{-\lambda}\lambda^{2}}{2!} = 9\frac{e^{-\lambda}\lambda^{4}}{4!} + 90\frac{e^{-\lambda}\lambda^{6}}{6!}$$

Dividing by $e^{-\lambda}\lambda^{2}$, we get

$$\frac{1}{2!} = \frac{9\lambda^{2}}{4!} + \frac{90\lambda^{4}}{6!} \Rightarrow \frac{1}{2} = \frac{3}{8}\lambda^{2} + \frac{1}{8}\lambda^{4} \Rightarrow \lambda^{4} + 3\lambda^{2} - 4 = 0$$

$$\Rightarrow (\lambda^{2}+4)(\lambda^{2}-1) = 0 \Rightarrow \lambda^{2} = -4 \text{ (or) } \lambda^{2} = 1 \Rightarrow \lambda = 1$$

$$(\because \lambda > 0)$$

For a poisson distribution, $Var(X) = \lambda = 1$

⑥ If X & Y are independent poisson variate such that $P(X=1) = P(X=2)$ & $P(Y=2) = P(Y=3)$ find the variance of $X - 2Y$.

Sol: WKT $P(X=x) = \frac{e^{-\lambda}\lambda^{x}}{x!}$

Given $P(X=1) = P(X=2) \Rightarrow \frac{e^{-\lambda}\lambda}{1!} = \frac{e^{-\lambda}\lambda^{2}}{2!} \Rightarrow \lambda = 2$

Also given $P(Y=2) = P(Y=3) \Rightarrow \frac{e^{-\mu}\mu^{2}}{2!} = \frac{e^{-\mu}\mu^{3}}{3!} \Rightarrow \mu = 3$

$Var(X) = 2 = \lambda$, $Var(Y) = \mu = 3$

$\therefore Var(X-2Y) = Var(X) + (-2)^{2}Var(Y) = 2 + 4(3) = 14$

⑦ If X & Y are independent Poisson variates with means λ_1 & λ_2 respectively, find the probability that (i) $X+Y=k$, (ii) $X=Y$.

Sol: (i) WKT for a Poisson variate 'X'

$$P(X=k) = \frac{e^{-\lambda}\lambda^{k}}{k!}, \quad k = 0, 1, 2, \dots$$

$$\therefore P(X+Y=k) = \frac{e^{-(\lambda_1+\lambda_2)} \times (\lambda_1+\lambda_2)^{k}}{k!}$$ (By additive property of Poisson distribution)

(ii) $P(X=Y) = \sum_{r=0}^{\infty} P(X=r \cap Y=r) = \sum_{r=0}^{\infty} P(X=r) \cdot P(Y=r)$ ($\because$ X & Y are independent)

$$= \sum_{r=0}^{\infty} \frac{e^{-\lambda_1}\lambda_1^{r}}{r!} \cdot \frac{e^{-\lambda_2}\lambda_2^{r}}{r!} = e^{-(\lambda_1+\lambda_2)} \sum_{r=0}^{\infty} \frac{(\lambda_1\lambda_2)^{r}}{(r!)^{2}}$$

⑧ The manufacturer of pins knows that 2% of his products are defective. If he sells pins in boxes of 100 & guarantees that not more than 4 pins will be defective. What is the probability that a box will fail to meet the guaranteed quality?

Sol: Given $n=100$, $p=2\% = \frac{2}{100} = 0.02$

$$\text{Mean } \lambda = np = 100 \times 0.02 = 2$$

The poisson distribution is $P(X=x) = \frac{e^{-\lambda}\lambda^x}{x!} = \frac{e^{-2}2^x}{x!}$

Now P(a box will fail to meet the guaranteed quality) $= P(X>4)$

$$= 1-P(X\leq 4) = 1-[P(X=0)+P(X=1)+P(X=2)+P(X=3)+P(X=4)]$$

$$= 1-\left[\frac{e^{-2}2^0}{0!}+\frac{e^{-2}2^1}{1!}+\frac{e^{-2}2^2}{2!}+\frac{e^{-2}2^3}{3!}+\frac{e^{-2}2^4}{4!}\right]$$

$$= 1-e^{-2}\left[1+2+2+\frac{4}{3}+\frac{2}{3}\right] = 1-e^{-2}(7) = 0.0527$$

⑨ If X & Y are independent poisson r.v., show that the conditional distribution of X given X+Y is a binomial distribution.

Sol: Let X & Y are independent poisson R.V.'s with parameters λ_1 & λ_2 respectively.

Now $P(X=r/X+Y=n) = \frac{P(X=r \text{ and } X+Y=n)}{P(X+Y=n)}$ $\# \; \cancel{P(X+Y)}.P($

$$= \frac{P(X=r \text{ and } Y=n-r)}{P(X+Y=n)} = \frac{P(X=r).P(Y=n-r)}{P(X+Y=n)}$$

$$= \frac{\frac{e^{-\lambda_1}\lambda_1^r}{r!} \cdot \frac{e^{-\lambda_2}\lambda_2^{n-r}}{(n-r)!}}{\frac{e^{-(\lambda_1+\lambda_2)}(\lambda_1+\lambda_2)^n}{n!}}$$

[∵ X is a poisson variate with parameter λ_1, Y is a poisson variate with parameter λ_2, X+Y is a poisson variate with parameter $\lambda_1+\lambda_2$]

$$= \frac{n!}{r!(n-r)!}\left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^r\left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)^{n-r}$$

$$= nC_r \, p^r q^{n-r} \text{ where } p = \frac{\lambda_1}{\lambda_1+\lambda_2} \; \& \; q = \frac{\lambda_2}{\lambda_1+\lambda_2}$$

which is a p.d.f. of a binomial distribution.

⑩ The sum of two independent Poisson variates is a Poisson variate.

Sol: Let X_1, X_2 be the two independent Poisson variate with parameter λ_1, λ_2 respectively.

Now, $M_{X_1+X_2}(t) = M_{X_1}(t)\, M_{X_2}(t) = e^{\lambda_1(e^t-1)}\, e^{\lambda_2(e^t-1)}$

$= e^{(\lambda_1+\lambda_2)(e^t-1)} = e^{\lambda(e^t-1)}$

∴ The sum of 2 independent poisson variates is a poisson variate.

---

⑪ If X_1 & X_2 are independent poisson variates show that $X_1 - X_2$ is not a poisson variate.

Sol: ~~Let~~ Given X_1 & X_2 ~~be~~ are the two independent Poisson variates with parameter λ_1, λ_2 respectively.

Now $M_{X_1-X_2}(t) = M_{X_1}(t) . M_{-X_2}(t) = M_{X_1}(t) . M_{X_2}(-t)$

$= e^{\lambda_1(e^t-1)} . e^{\lambda_2(e^{-t}-1)}$ which cannot be expressed in the form of $e^{\lambda(e^t-1)}$

∴ $X_1 - X_2$ is not a poisson variate.

Geometric distribution:

A r.v. X is said to follow Geometric distribution, if it assumes only non-(-)ve values & its probability mass func. is given by

$P(X=x) = (1-p)^{x-1} p = q^{x-1} p \quad , x = 1, 2, \dots , \quad 0 < p \le 1.$

The m.g.f. $M_X(t) = \sum_{x=1}^{\infty} e^{tx} p(x) = \sum_{x=1}^{\infty} e^{tx} q^{x-1} p = \sum_{x=1}^{\infty} pe^t (qe^t)^{x-1}$

$= pe^t \sum_{x=1}^{\infty} (qe^t)^{x-1} = pe^t [1 + qe^t + (qe^t)^2 + \dots]$

$= pe^t [1 - qe^t]^{-1} = \frac{pe^t}{1-qe^t} = \frac{p}{e^{-t}-q}$

Mean $E(X) = \left[\frac{d}{dt} M_X(t)\right]_{t=0} = \left[\frac{d}{dt}\left(\frac{p}{e^{-t}-q}\right)\right]_{t=0}$

$= \left[\frac{(e^{-t}-q).0 - p(e^{-t}.-1)}{(e^{-t}-q)^2}\right]_{t=0} = \frac{p}{(1-q)^2} = \frac{1}{p}$

$E(X^2) = \left[\frac{d^2}{dt^2} M_X(t)\right]_{t=0} = \left[\frac{d}{dt}\left(\frac{pe^{-t}}{(e^{-t}-q)^2}\right)\right]_{t=0}$

$= \left[\frac{(e^{-t}-q)^2\, pe^{-t}.-1 - pe^{-t}.2(e^{-t}-q).e^{-t}.-1}{(e^{-t}-q)^4}\right]_{t=0}$

$= \frac{-p^3 + 2p(1-q)}{(1-q)^4} = \frac{-p^3+2p^2}{p^4} = \frac{-p+2}{p^2} = \frac{-1}{p} + \frac{2}{p^2}$

$Var(X) = E(X^2) - [E(X)]^2 = \frac{-1}{p} + \frac{2}{p^2} - \frac{1}{p^2} = \frac{-1}{p} + \frac{1}{p^2} = \frac{-p+1}{p^2} = \frac{q}{p^2}$

Problems:

① If the probability that a target is destroyed on any one shot is 0.5, what is the probability that it would be destroyed on 6th attempt.

Sol: Given $p = 0.5$

$q = 1 - 0.5 = 0.5$

WKT $P(X=x) = q^{x-1}p$

$P(X=6) = q^5 p = (0.5)^5 (0.5) = 0.0156$

② If the probability is 0.05 that a certain kind measuring device will show excessive drift, what is the probability that the sixth of these measuring devices tested will be the first to show excessive drift?

Sol: Here $p = 0.05$, $q = 1-p = 1-0.05 = 0.95$, $x = 6$

WKT $P(X=x) = q^{x-1}p = (0.95)^5 (0.05) = 0.0387$

③ Let one copy of a magazine out of 10 copies bears a special prize following geometric random distribution. Determine its mean & variance.

Sol: Given $p = \frac{1}{10}$, $q = 1-p = 1-\frac{1}{10} = \frac{9}{10}$

Mean of the geometric distribution is $= \frac{1}{p} = 10$

Variance $= \frac{q}{p^2} = \frac{9/10}{(1/10)^2} = \frac{9}{10} \times 10^2 = 90$

④ Suppose that a trainee soldier shoots a target in an independent fashion. If the probability that the target is shot on any one shot is 0.8.

(i) What is the probability that the target would be hit on 6th attempt?

(ii) What is the probability that it takes him less than 5 shots?

(iii) What is the probability that it takes him an even no. of shots?

Sol: Given $p = 0.8$, $q = 1-p = 1-0.8 = 0.2$

The geometric distribution is $P(X=x) = q^{x-1}p$, $x = 1, 2, \ldots$

(i) P(the target would be hit on the 6th attempt) $= P(x=6)$

$= (0.2)^5 (0.8) = 0.000256$

(ii) P(it takes him less than 5 shots) $= P(x<5) = P(x=1) + P(x=2) + P(x=3) + P(x=4)$

$= (0.2)^0(0.8) + (0.2)^1(0.8) + (0.2)^2(0.8) + (0.2)^3(0.8)$

$= (0.8) + (0.2 \times 0.8) + (0.2)^2(0.8) + (0.2)^3(0.8)$

$= 0.9984$

(iii) P(it takes him an even no. of shots) $= P(X=2)+P(X=4)+P(X=6)+\cdots$

$$= (0.2)^{2-1}(0.8) + (0.2)^{4-1}(0.8) + (0.2)^{6-1}(0.8) + \cdots$$

$$= (0.2)(0.8) + (0.2)^3(0.8) + (0.2)^5(0.8) + \cdots$$

$$= (0.2)(0.8)\left[1+(0.2)^2+(0.2)^4+\cdots\right] = 0.16\left[1+0.04+(0.04)^2+\cdots\right]$$

$$= 0.16\left[1-0.04\right]^{-1} = 0.16\left[0.96\right]^{-1} = \frac{0.16}{0.96} = 0.1667$$

⑮ Establish the memoryless property of geometric distribution.

Sol: If X has a geometric distribution, then for any two positive integers m & n, $P[X>m+n / X>m] = P(X>n)$

Proof: $P[X>m+n / X>m] = \dfrac{P[X>m+n \cap X>m]}{P[X>m]} = \dfrac{P[X>m+n]}{P[X>m]}$

Taking $P[X=r] = q^{r-1}p,\ r=1,2,3,\ldots$

$$P[X>k] = \sum_{r=k+1}^{\infty} q^{r-1}p = q^k p + q^{k+1}p + q^{k+2}p + \cdots$$

$$= q^k p\left[1+q+q^2+\cdots\right] = q^k p\left[1-q\right]^{-1} = q^k \quad (\because 1-q=p)$$

$\therefore P[X>m+n] = q^{m+n}$ & $P[X>m] = q^m$

$$\therefore P[X>m+n / X>m] = \frac{q^{m+n}}{q^m} = q^n$$

$$P[X>n] = q^n$$

Hence $P[X>m+n / X>m] = P[X>n]$

Uniform Distribution (or) Rectangular Distribution :

The p.d.f. of a uniform variable X in $(-a,a)$ is given by

$$f(x) = \begin{cases} \dfrac{1}{2a}, & -a < x < a \\ 0, & \text{otherwise} \end{cases}$$

Derivation: $f(x) = \dfrac{1}{b-a}, \quad a < x < b$

The m.g.f. $M_X(t) = \displaystyle\int_a^b e^{tx} f(x)\,dx = \int_a^b e^{tx} \cdot \frac{1}{b-a}\,dx = \frac{1}{b-a}\left[\frac{e^{tx}}{t}\right]_a^b$

$$= \frac{1}{(b-a)t}\left[e^{tb} - e^{ta}\right]$$

$$= \frac{\left[1 + \frac{bt}{1!} + \frac{(bt)^2}{2!} + \cdots\right] - \left[1 + \frac{at}{1!} + \frac{(at)^2}{2!} + \cdots\right]}{(b-a)t}$$

$$= \frac{\frac{(b-a)t}{1!} + \frac{(b^2-a^2)t^2}{2!} + \frac{(b^3-a^3)t^3}{3!} + \cdots}{(b-a)t}$$

$$= 1 + \frac{(b+a)t}{2!} + \frac{(b^2+ba+a^2)}{3!}t^2 + \cdots$$

$$\text{Mean} = E(X) = \left[\frac{d}{dt} M_x(t)\right]_{t=0}$$

$$= \left[\frac{d}{dt}\left(1 + \frac{(b+a)t}{2!} + \frac{(b^2+ba+a^2)t^2}{3!} + \cdots\right)\right]_{t=0}$$

$$= \left[\frac{b+a}{2} + \frac{(b^2+ba+a^2)2t}{3!} + \cdots\right]_{t=0} = \frac{b+a}{2}$$

$$E[x^2] = \left[\frac{d^2}{dt^2} M_x(t)\right]_{t=0} = \left[\frac{d}{dt}\left(\frac{b+a}{2} + \frac{(b^2+ba+a^2)2t}{6} + \cdots\right)\right]_{t=0}$$

$$= \left[\frac{b^2+ba+a^2}{3} + \cdots\right]_{t=0} = \frac{1}{3}(b^2+ba+a^2)$$

$$\text{Var}(x) = E(x^2) - [E(x)]^2 = \frac{1}{3}(b^2+ba+a^2) - \left(\frac{b+a}{2}\right)^2$$

$$= \frac{1}{3}(b^2+ba+a^2) - \frac{1}{4}(a^2+b^2+2ab) = \frac{4b^2+4ab+4a^2-3a^2-3b^2-6ab}{12}$$

$$= \frac{1}{12}(a^2+b^2-2ab) = \frac{1}{12}(b-a)^2$$

Problems:

① Electric trains on a certain line run every half an hour between mid-night & six in the morning. What is the probability that a man entering the station at a random time during this period will have to wait atleast 20 minutes?

Sol: Let x be the r.v. which denotes the waiting time for the next train. Assume that a man arrives at the station at random, the r.v. X is distributed uniformly in (0, 30) with p.d.f. $f(x) = \begin{cases} \frac{1}{30} & , 0 < x < 30 \\ 0, & \text{otherwise} \end{cases}$

$$\therefore P(\text{atleast 20 minutes}) = P(x \geq 20) = \int_{20}^{30} f(x)\,dx$$

$$= \frac{1}{30}\int_{20}^{30} dx = \frac{1}{30}(x)_{20}^{30} = \frac{1}{30}(30-20) = \frac{10}{30} = \frac{1}{3}$$

② S.T. for the uniform distribution $f(x) = \begin{cases} \frac{1}{2a} & , -a < x < a \\ 0, & \text{elsewhere} \end{cases}$ the m.g.f. about the origin is $\frac{\sinh at}{at}$. Also, moments of even order are given by $\mu_{2n} = \frac{a^{2n}}{2n+1}$.

Sol: WKT the m.g.f. of uniform distribution in the interval (a, b) is

$$M_x(t) = \int_a^b e^{tx} f(x)\,dx$$

Here $f(x) = \frac{1}{2a}$ in $-a < x < a$

$$\therefore M_x(t) = \int_{-a}^{a} e^{tx} \frac{1}{2a}\, dx = \frac{1}{2a}\left(\frac{e^{tx}}{t}\right)_{-a}^{a} = \frac{1}{2at}\left(e^{ta} - e^{-ta}\right)$$

$$= \frac{1}{at} \sinh at \quad \left(\because \frac{e^{x} - e^{-x}}{2} = \sinh x\right)$$

$$M_x(t) = \frac{1}{at} \sinh at = \frac{1}{at}\left[at + \frac{(at)^3}{3!} + \dots\right] = 1 + \frac{(at)^2}{3!} + \frac{(at)^4}{5!} + \dots$$

Since there are no terms with odd powers of t in $M_x(t)$ all moments of odd order about origin vanish. (i.e) $\mu'_{2n+1} = 0$

In particular $\mu'_1 = 0 \Rightarrow$ Mean $= 0$

Thus $\mu_{2n} = \mu'_{2n}$ ($\because$ mean $= 0$)

$\therefore \mu_{2n+1} = 0, \ n = 0, 1, 2, \dots$

(ii) All moments of odd order about mean vanish. The moments of even order are given by $\mu_{2n} =$ coefficient of $\frac{t^{2n}}{(2n)!}$ in $M_x(t) = \frac{a^{2n}}{(2n+1)}$

---

③ If X is a r.v. uniformly distributed in (0, 1), find the p.d.f. of $Y = \sin X$. Also find the mean & variance of Y.

Sol: Given $Y = \sin X$. X has a uniform p.d.f. over (0, 1).

$$G(y) = P(Y \le y) = P(\sin X \le y) = P(X \le \sin^{-1} y) = \int_0^{\sin^{-1} y} dx$$

$$= [x]_0^{\sin^{-1} y} = \sin^{-1} y$$

$$g(y) = \frac{d}{dy}(G(y)) = \frac{d}{dy}(\sin^{-1} y) = \frac{1}{\sqrt{1 - y^2}}$$

$$\text{Mean} = E(Y) = \int_0^1 \sin x\, dx = [-\cos x]_0^1 = -\cos 1 + \cos 0 = -0.5403 + 1 = 0.4597$$

$$E(Y^2) = \int_0^1 \sin^2 x\, dx = \int_0^1 \frac{1 - \cos 2x}{2}\, dx = \frac{1}{2}\left[x - \frac{\sin 2x}{2}\right]_0^1$$

$$= \frac{1}{2}\left[1 - \frac{\sin 2}{2}\right] = \frac{1}{2} - \frac{\sin 2}{4} = 0.2727$$

~~$Var(Y) = E(Y^2) - [E(Y)]^2 = \frac{1}{2} - \frac{\sin 2}{4} + (0.4597)^2 [1 - \cos 1]^2$~~

~~$= \frac{1}{2} - \frac{\sin 2}{4} - 1 - \cos^2 1 + 2\cos 1 = -\frac{1}{2}$~~

$$Var(Y) = E(Y^2) - [E(Y)]^2 = 0.2727 - (0.4597)^2 = 0.0614$$

---

④ X is uniformly distributed with mean 1 & variance $\frac{4}{3}$, find $P(X < 0)$

Sol: Given that mean $= 1 \Rightarrow \frac{b + a}{2} = 1 \Rightarrow a + b = 2$ —①

Variance $= \frac{4}{3} \Rightarrow \frac{\cancel{V(X)}}{\cancel{V}} \; \frac{(a-b)^2}{12} = 4/3 \Rightarrow (a-b)^2 = 16$

$\Rightarrow a - b = 4$ —— ②

Solving ① & ②, $2a = 6 \Rightarrow a = 3$, $b = -1$

$\therefore f(x) = \frac{1}{b-a} = \frac{1}{-1-3} = \frac{-1}{4}$

$$P(x<0) = \int_{-1}^{0} \frac{-1}{4} dx = \frac{-1}{4} (x)_{-1}^{0}$$

---

⑤ A r.v. X has a uniform distribution over $(-3,3)$ compute (i) $P(x<2)$, $P(|x|<2)$, $P(|x-2|<2)$ (ii) Find K for which $P(x>K) = \frac{1}{3}$.

Sol: WKT the p.d.f. of a r.v. X which is distributed uniformly in $(-a,a)$ is $f(x) = \begin{cases} \frac{1}{2a} & , -a<x<a \\ 0 & , \text{otherwise} \end{cases}$

Here $a = 3$

$\therefore$ P.d.f. is $f(x) = \begin{cases} \frac{1}{6} & , -3<x<3 \\ 0 & , \text{otherwise} \end{cases}$

(i) $P(x<2) = \int_{-3}^{2} f(x)dx = \frac{1}{6}\int_{-3}^{2} dx = \frac{1}{6}(x)_{-3}^{2} = \frac{1}{6}(2+3) = \frac{5}{6}$

$P(|x|<2) = P(-2<x<2) = \int_{-2}^{2} f(x)dx = \frac{1}{6}(x)_{-2}^{2} = \frac{1}{6}(2+2) = \frac{2}{3}$

$P(|x-2|<2) = P(-2<x-2<2) = P(0<x<4) = \int_{0}^{3} f(x)dx$

$= \frac{1}{6}(x)_{0}^{3} = \frac{1}{6}(3) = \frac{1}{2}$

(ii) Given $P(x>K) = \frac{1}{3} \Rightarrow \int_{K}^{3} f(x)dx = \frac{1}{3} \Rightarrow \frac{1}{6}(x)_{K}^{3} = \frac{1}{3}$

$\Rightarrow 3 - K = 2 \Rightarrow K = 1$

---

⑥ Buses arrive at a specified bus stop at 15 minutes intervals starting at 7a.m. that is 7a.m., 7.15a.m., 7.30a.m., etc. If a passenger arrives at the bus stop at a random time which is uniformly distributed between 7 & 7.30a.m. find the probability that he waits (a) less than 5 minutes (b) atleast 12 minutes for a bus.

Sol: Let X denote the no. of minutes past 7 that the passenger arrives at the bus stop. In the interval $(0,30)$ X is a uniform r.v. & it follows that a passenger will have to wait less than 5 minutes if he arrives between

7.10 & 7.15 or between 7.25 & 7.30.

The p.d.f. is $f(x) = \begin{cases} \frac{1}{30}, & 0 < x < 30 \\ 0, & \text{otherwise} \end{cases}$

(a) $P(10 \leq X \leq 15) + P(25 \leq X \leq 30) = \int_{10}^{15} \frac{1}{30}\,dx + \int_{25}^{30} \frac{1}{30}\,dx$

$$= \frac{1}{30}\left[15 - 10 + 30 - 25\right] = \frac{10}{30} = \frac{1}{3}$$

(b) Passenger waits atleast 12 minutes (i.e.) he arrives between 7-7.03 or 7.15-7.18.

$P(\text{waiting time atleast 12 minutes}) = P(0 \leq X \leq 3) + P(15 \leq X \leq 18)$

$$= \int_0^3 \frac{1}{30}\,dx + \int_{15}^{18} \frac{1}{30}\,dx = \frac{1}{30}\left[3 - 0 + 18 - 15\right] = \frac{6}{30} = \frac{1}{5}$$

Exponential Distribution:

A continuous r.v. X is said to follow exponential distribution if its p.d.f. is given by, $f(x) = \begin{cases} \alpha e^{-\alpha x}, & x \geq 0, \alpha > 0 \\ 0, & \text{otherwise} \end{cases}$

Derivation: $f(x) = \lambda e^{-\lambda x}, x \geq 0, \lambda > 0$

The m.g.f. $M_X(t) = \int_{-\infty}^{\infty} e^{tx} f(x)\,dx = \int_0^{\infty} e^{tx} \lambda e^{-\lambda x}\,dx = \lambda \int_0^{\infty} e^{(t-\lambda)x}\,dx$

$$= \lambda\left[\frac{e^{(t-\lambda)x}}{t-\lambda}\right]_0^{\infty} = \left(\frac{\lambda}{t-\lambda}\left[e^{-(\lambda-t)x}\right]_0^{\infty}\right)$$

$$= \frac{\lambda}{t-\lambda}\left[0 - 1\right] = \frac{\lambda}{\lambda - t}$$

Mean $E(X) = \left[\frac{d}{dt} M_X(t)\right]_{t=0} = \left[\frac{d}{dt}\left(\frac{\lambda}{\lambda-t}\right)\right]_{t=0} \neq \cancel{\lambda\left[\frac{(\lambda-t)^{-2}}{-2}\right]}$

$$= \lambda\left[-1(\lambda-t)^{-2}(-1)\right]_{t=0} = \frac{\lambda}{\lambda^2} = \frac{1}{\lambda}$$

$E(X^2) = \left[\frac{d^2}{dt^2} M_X(t)\right]_{t=0} = \left[\frac{d}{dt}\left(\frac{\lambda}{(\lambda-t)^2}\right)\right]_{t=0} = \lambda\left[(-2)(\lambda-t)^{-3}(-1)\right]_{t=0}$

$$= \frac{2\lambda}{\lambda^3} = \frac{2}{\lambda^2}$$

$\text{Var}(X) = E(X^2) - [E(X)]^2 = \frac{2}{\lambda^2} - \left(\frac{1}{\lambda}\right)^2 = \frac{1}{\lambda^2}$

Memoryless property of exponential distribution:

If X is exponentially distributed, then $P(X > s+t \,/\, X > s) = P(X > t)$, for any $s, t > 0$.

Proof: $P(X > k) = \int_k^{\infty} \lambda e^{-\lambda x}\,dx = \lambda\left[\frac{e^{-\lambda x}}{-\lambda}\right]_k^{\infty} = -\left[0 - e^{-\lambda k}\right] = e^{-\lambda k}$ —— ①

Also, $P(X>s+t/X>s) = \dfrac{P(X>s+t \text{ and } X>s)}{P(X>s)}$

$$= \frac{P(X>s+t)}{P(X>s)} = \frac{e^{-\lambda(s+t)}}{e^{-\lambda s}} \text{ (by ①)} = e^{-\lambda t} = P(X>t)$$

Hence $P(X>s+t/X>s) = P(X>t)$

Note: The converse of this result is also true. (ie) If $P(X>s+t/X>s) = P(X>t)$ then X follows an exponential distribution

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Problems:

① The length of time a person speaks over phone follows exponential distribution with parameter $\frac{1}{6}$. What is the probability that the person will talk for (i) more than 8 minutes (ii) between 4 & 8 minutes?

Sol: Given $f(x) = \frac{1}{6} e^{-x/6}$

$$\text{(i) } P[X>8] = \int_8^\infty f(x)\,dx = \frac{1}{6}\int_8^\infty e^{-x/6}\,dx = \frac{1}{6}\left[\frac{e^{-x/6}}{-1/6}\right]_8^\infty$$

$$= -\left[0 - e^{-4/3}\right] = e^{-4/3} = 0.2636$$

$$\text{(ii) } P(4 \le X \le 8) = \int_4^8 \frac{1}{6} e^{-x/6}\,dx = \frac{1}{6}\left[\frac{e^{-x/6}}{-1/6}\right]_4^8 = -\left[e^{-4/3} - e^{-2/3}\right]$$

$$= -\left[0.2636 - 0.5134\right] = 0.2498$$

---

② If X has an exponential distribution with parameter α, find the p.d.f. of $Y = \log X$.

Sol: $f_X(x) = \alpha e^{-\alpha y}$

$f_Y(y) = \frac{d}{dy} F_Y(y)$

$F_Y(y) = P(Y \le y) = P(\log X \le y) = P(X \le e^y) = e^{-\alpha e^y}$

$$f_Y(y) = \frac{d}{dy}\left[e^{-\alpha e^y}\right] = e^{-\alpha e^y}\left[-\alpha e^y\right] = -\alpha e^y e^{-\alpha e^y}$$

$$= \alpha e^y e^{-\alpha e^y}, \quad -\infty < y < \infty$$

---

③ The time in hours required to repair a machine is exponentially distributed with parameter $\lambda = \frac{1}{2}$. (i) What is the probability that the repair time exceeds 2h? (ii) What is the conditional probability that a repair takes atleast 10h given that its duration exceeds 9h?

Sol: Given $\lambda = \frac{1}{2}$. Let X represents the time to repair the machine.

Then the density funt. of X is given by $f(x) = \lambda e^{-\lambda x} = \frac{1}{2} e^{-x/2}, x \geq$

(i) $P(X>2) = \int_2^\infty \frac{1}{2} e^{-x/2} dx = \frac{1}{2}\left[\frac{e^{-x/2}}{-1/2}\right]_2^\infty = -\left[0-e^{-1}\right] = e^{-1} = 0.3679$

(ii) The conditional probability that a repair takes atleast 10h given that its duration exceeds 9h is given by,

$P(X>10 \mid X>9) = P(X>9+1 \mid X>9) = P(X>1)$

$(\because P(X>s+t \mid X>s) = P(X>t))$ by memoryless property

$= \int_1^\infty \frac{1}{2} e^{-\frac{1}{2}x} dx$

$= \frac{1}{2}\left[\frac{e^{-x/2}}{-1/2}\right]_1^\infty = -\left[0-e^{-1/2}\right]$

$= e^{-1/2} = 0.6065$

④ If a continuous r.v. X follows uniform distribution in the interval $(0,2)$ & a continuous r.v. Y follows exponential distribution with parameter α, find α such that $P(X<1) = P(Y<1)$.

Sol: Since X follows uniform distribution over $(0,2)$, we get

$$f(x) = \begin{cases} \frac{1}{2-0}, & 0<x<2 \\ 0, & \text{otherwise} \end{cases} = \begin{cases} \frac{1}{2}, & 0<x<2 \\ 0, & \text{otherwise} \end{cases}$$

Y follows exponential distribution $\therefore f(y) = \alpha e^{-\alpha y}, \; y \geq 0$

Given $P(X<1) = P(Y<1) \Rightarrow \int_0^1 f(x)dx = \int_0^1 f(y)dy$

$\Rightarrow \int_0^1 \frac{1}{2} dx = \int_0^1 \alpha e^{-\alpha y} dy \Rightarrow \frac{1}{2}(x)_0^1 = \alpha\left(\frac{e^{-\alpha y}}{-\alpha}\right)_0^1$

$\Rightarrow \frac{1}{2} = -(e^{-\alpha}-1) \Rightarrow \frac{1}{2} = (1-e^{-\alpha})$

$\Rightarrow e^{-\alpha} = \frac{1}{2} \Rightarrow -\alpha = \log_e 1/2 = \log_e 1 - \log_e 2 = 0 - \log_e 2$

$\Rightarrow \alpha = \log_e 2 = 0.6931$

Normal Distribution:

① In a Normal distribution whose mean is 12 & standard deviation is 2.

existing in a high degree

Find the probability intense from $X = 9.6$ to $X = 13.8$.

Sol: Given $\mu = 12$, $\sigma = 2$

Here $z = \frac{X-\mu}{\sigma}$

$$P(9.6 \le X \le 13.8) = P\left(\frac{9.6-12}{2} \le \frac{X-\mu}{\sigma} \le \frac{13.8-12}{2}\right)$$

$$= P(-1.2 \le Z \le 0.9)$$

$$= P(-1.2 \le Z \le 0) + P(0 \le Z \le 0.9)$$

$$= P(0 \le Z \le 1.2) + P(0 \le Z \le 0.9)$$

$$= 0.3849 + 0.3159 = 0.7008$$

② The savings bank account of a customer showed an average balance of Rs.150 & a S.D. of Rs.50. Assuming that the account balances are normally distributed. (i) What percentage of account is over Rs.200? (ii) What percentage of account is between Rs.120 & Rs.170? (iii) What percentage of account is less than Rs.75?

Sol: Given $\mu = 150$ & $\sigma = 50$

(i) $P(X > 200) = P\left(\frac{X-\mu}{\sigma} > \frac{200-150}{50}\right) = P(Z > 1) = 0.5 - P(0 < Z < 1)$

$= 0.5 - 0.3413 = 0.1587$ ∴ Percentage of account is over Rs.200 is 15.87%.

(ii) $P(120 < X < 170) = P\left(\frac{120-150}{50} < Z < \frac{170-150}{50}\right)$

$$= P(-0.6 < Z < 0.4) = P(-0.6 < Z < 0) + P(0 < Z < 0.4)$$

$$= P(0 < Z < 0.6) + P(0 < Z < 0.4)$$

$$= 0.2257 + 0.1554 = 0.3811$$

∴ Percentage of account is between Rs.120 & Rs.170 is 38.11%.

(iii) $P(X < 75) = P\left(Z < \frac{75-150}{50}\right) = P(Z < -1.5) = 0.5 - P(-1.5 < Z < 0)$

$$= 0.5 - P(0 < Z < 1.5) = 0.5 - 0.4332 = 0.0668$$

∴ Percentage of account is less than Rs.75 is 6.68%.

③ For a normal distribution with mean 2 & variance 9, find the value of x_1 of the variable such that the probability of the variable lying in the interval $(2, x_1)$ is 0.4115.

<u>Sol:</u> Let x follows normal distribution with mean $\mu = 2$ & variance $\sigma^2 = 9$

Given $P(2 < X < x_1) = 0.4115$

$$\Rightarrow P\left(\frac{2-2}{3} < \frac{X-\mu}{\sigma} < \frac{x_1-2}{3}\right) = 0.4115$$

$$\Rightarrow P\left(0 < Z < \frac{x_1-2}{3}\right) = 0.4115$$

From the table values $P(0 < Z < 1.35) = 0.4115$, $\frac{x_1-2}{3} = 1.35$

$$\therefore x_1 = 6.05$$

④ If the actual amount of instant coffee which a filling machine puts into '6-ounce' jars is a random variable having a normal distribution with S.D = 0.05 ounce & if only 3% of the jars are to contain less than 6 ounces of coffee, what must be the mean fill of these jars?

<u>Sol:</u> Let x be the actual amount of coffee put into the jars.

Then x follows $N(\mu, \sigma)$ & $\sigma = 0.05$

Given that $P(x < 6) = 0.03$

$$\therefore P(-\infty < x < 6) = 0.03$$

$$\Rightarrow P\left(\frac{-\infty-\mu}{0.05} < \frac{x-\mu}{\sigma} < \frac{6-\mu}{0.05}\right) = 0.03$$

$$\Rightarrow P\left(-\infty < Z < \frac{6-\mu}{0.05}\right) = 0.03$$

$$\Rightarrow P\left(0 < Z < \frac{\mu-6}{0.05}\right) = 0.47$$

From the table values, $P(0 < Z < 1.88) = 0.47$

$$\therefore \frac{\mu-6}{0.05} = 1.88 \Rightarrow \mu = 6.094 \text{ ounces}$$

⑤ X is a normal variate with mean 30 & S.D. 5. Find the probabilities that
(i) $26 \le X \le 40$ (ii) $X \ge 45$ (iii) $|X-30| > 5$.

<u>Sol:</u> Given $\mu = 30$, $\sigma = 5$

(i) $P(26 \le X \le 40) = P\left(\frac{26-30}{5} \le Z \le \frac{40-30}{5}\right) = P(-0.8 \le Z \le 2)$

$$= P(0 \le z \le 0.8) + P(0 \le z \le 2)$$

$$= 0.2881 + 0.4772 = 0.7653$$

(ii) $$P(x \ge 45) = P\left(z \ge \frac{45-30}{5}\right) = P(z \ge 3) = 0.5 - P(0 \le z \le 3)$$

$$= 0.5 - 0.4987 = 0.0013$$

(iii) $$P(|x-30| > 5) = 1 - P(|x-30| \le 5)$$

$$= 1 - P(-5 \le x - 30 \le 5)$$

$$= 1 - P\left(\frac{-5}{5} \le z \le \frac{5}{5}\right) = 1 - P(-1 \le z \le 1)$$

$$= 1 - [P(0 \le z \le 1) + P(0 \le z \le 1)] = 1 - 2P(0 \le z \le 1)$$

$$= 1 - 2(0.3413) = 0.3174$$

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⑥ The marks obtained by a no. of students in a certain subject are approximately normally distributed with mean 65 & S.D. 5. If 3 students are selected at random from this group, what is the probability that atleast one of them would have scored above 75? [Given the area between $z=0$ & $z=2$ under the standard normal curve is 0.4772]

Sol: Given $\mu = 65$ & $\sigma = 5$ X - the mark obtained by a student.

$$P(x > 75) = P\left(\frac{x-\mu}{\sigma} > \frac{75-65}{5}\right) = P(z > 2)$$

$$= 0.5 - P(0 < z < 2) = 0.5 - 0.4772 = 0.0228$$

$p = P$ (a student scores above 75) $= 0.0228$

$q = 0.9772$ & $n = 3$

Y - no. of students scoring more than 75

$$f(y) = nc_y\, p^y q^{n-y} = 3c_y (0.0228)^y (0.9772)^{3-y}$$

$$P(Y \ge 1) = 1 - P(Y < 1) = 1 - P(Y = 0) = 1 - 3c_0 (0.0228)^0 (0.9772)^3$$

$$= 1 - (0.9772)^3 = 0.0669$$

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⑦ In an engineering examindion, a student is considered to have failed, secured second class, first class & distinction, according as he scores less than 45%, between 45% & 60%, between 60% & 75% & above 75% respectively. In a particular year 10% of the students failed in the examination & 5% of the students get distinction. Find the percentages of students who have got first class & second class. (Assume normal distribution of marks).

Sol: Let x represent the percentage of marks scored by the students in the examination.

Let x follows normal distribution.

Given $P(x<45)=0.1$ & $P(x>75)=0.05$

(ie) $P\left(-\infty < \frac{x-\mu}{\sigma} < \frac{45-\mu}{\sigma}\right)=0.1$ & $P\left(\frac{75-\mu}{\sigma} < \frac{x-\mu}{\sigma} < \infty\right)=0.05$

$\Rightarrow P\left(-\infty < z < \frac{45-\mu}{\sigma}\right)=0.1$ & $P\left(\frac{75-\mu}{\sigma} < z < \infty\right)=0.05$

$P\left(-\infty < z < \frac{45-\mu}{\sigma}\right)=0.1 \Rightarrow 0.5 - P\left(\frac{45-\mu}{\sigma} < z < 0\right)=0.1$

$\Rightarrow 0.5 - P\left(0 < z < \frac{\mu-45}{\sigma}\right)=0.1$

$\Rightarrow P\left(0 < z < \frac{\mu-45}{\sigma}\right)=0.4$

From the table values, $P(0<z<1.28)=0.4$

$\therefore \frac{\mu-45}{\sigma}=1.28 \Rightarrow \mu-45=1.28\sigma \Rightarrow \mu-1.28\sigma=45$ — ①

$P\left(\frac{75-\mu}{\sigma} < z < \infty\right)=0.05 \Rightarrow 0.5 - P\left(0<z<\frac{75-\mu}{\sigma}\right)=0.05$

$\Rightarrow P\left(0<z<\frac{75-\mu}{\sigma}\right)=0.5-0.05=0.45$

From the table values, $P(0<z<1.64)=0.45$

$\therefore \frac{75-\mu}{\sigma}=1.64 \Rightarrow 75-\mu=1.64\sigma \Rightarrow -\mu-1.64\sigma=-75$ — ②

① + ② $\Rightarrow -2.92\sigma=-30 \Rightarrow \sigma=10.27$

$\therefore \mu=58.15$

Now P(a student gets first class) $= P(60<x<75)$

$= P\left(\frac{60-58.15}{10.27} < z < \frac{75-58.15}{10.27}\right) = P(0.18<z<1.64)$

$= P(0<z<1.64) - P(0<z<0.18)$

$= 0.4495 - 0.0714 = 0.3781$

$\therefore$ Percentage of students getting first class = 38%. (appl.)

Now percentage of students getting second class

$= 100-(10+38+5) = 47$ (appl.)

TWO DIMENSIONAL RANDOM VARIABLES

Two dimensional random variable:

Let S be the sample space. Let $X = X(s)$ & $Y = Y(s)$ be two funs/. each assigning a real no/. to each outcome $s \in S$. Then (X,Y) is a two-dimensional random variable.

Two-dimensional discrete random variables:

If the possible values of (X,Y) are finite or countably infinite, then (X,Y) is called a two-dimensional discrete random variable. When (X,Y) is a two-dimensional discrete random variable the possible values of (X,Y) may be represented as (x_i, y_j), $i = 1,2,\dots,n$, $j = 1,2,\dots,m$.

Two-dimensional continuous random variables:

If (X,Y) can assume all values in a specified region R in the XY plane (X,Y) is called a two-dimensional continuous random variable.

Joint probability distribution:

The probabilities of the two events $A = \{X \le x\}$ & $B = \{Y \le y\}$ have defined as funs/. of x & y, respectively, called probability distribution funs/..

$F_X(x) = P(X \le x)$; $F_Y(y) = P(Y \le y)$

Joint probability distribution of two random variables X & Y:

The probability of the joint event $\{X \le x, Y \le y\}$, which is a fun/. of the nos/. x & y, by a joint probability distribution fun/. & denote it by the symbol $F_{X,Y}(x,y)$. Hence $F_{X,Y}(x,y) = P(X \le x, Y \le y)$.

Properties of the joint distribution:

① $F_{X,Y}(-\infty,-\infty) = 0$; $F_{X,Y}(-\infty, y) = 0$ & $F_{X,Y}(x,-\infty) = 0$

② $F_{X,Y}(\infty,\infty) = 1$

③ $0 \le F_{X,Y}(x,y) \le 1$

④ $F_{X,Y}(x,y)$ is a non-decreasing fun/. of x & y.

⑤ $F_{X,Y}(x_2,y_2) - F_{X,Y}(x_1,y_2) + F_{X,Y}(x_1,y_1) - F_{X,Y}(x_2,y_1)$

$$= P\{x_1 < X \le x_2 ; y_1 < Y \le y_2\} \ge 0$$

⑥ $F_{X,Y}(x,\infty) = F_X(x)$ & $F_{X,Y}(\infty,y) = F_Y(y)$

For a given fun/. to be a valid joint distribution fun/. of two dimensional RVs X & Y, it must satisfy the properties ①, ② & ⑤.

Joint probability fun/. of the discrete random variables X & Y:

If (X,Y) is a two-dimensional discrete r.v. such that $f(x_i,y_j) = P(X=x_i, Y=y_j) = P_{ij}$ is called the joint probability fun/. or joint probability mass fun/. of (X,Y) provided the following conditions are satisfied. (i) $P_{ij} \geq 0, \forall\, i \,\&\, j$ (ii) $\sum_j \sum_i P_{ij} = 1$

The set of triples $\{x_i, y_j, P_{ij}\}$, $i=1,2,\dots,n$, $j=1,2,\dots,m$ is called the joint probability distribution of (X,Y).

Marginal probability distribution:

The individual probability distribution of a random variable is referred to as its marginal probability distribution. In general, the marginal probability distribution of X can be determined from the joint probability distribution of X & other random variables.

Marginal probability mass fun/. of X:

If the joint probability distribution of two random variables X & Y is given, then the marginal probability fun/. of X is given by

$$f(x) = P_X(x_i) = P(X=x_i)$$

$$= P[X=x_i, Y=y_1] + P[X=x_i, Y=y_2] + \dots + P[X=x_i, Y=y_j] + \dots + P[X=x_i, Y=y_m]$$

$$= P_{i1} + P_{i2} + \dots + P_{ij} + \dots + P_{im} = \sum_{j=1}^{m} P_{ij} = \sum_{j=1}^{m} p(x_i, y_j) = P_{i.}$$

Note: The set $\{x_i, P_{i.}\}$ is called the marginal distribution of X.

Marginal probability mass fun/. of Y:

If the joint probability distribution of two random variables X & Y is given, then the marginal probability fun/. of Y is given by

$$f(y) = P_Y(y_j) = P(Y=y_j) = P_{.j}$$

Here $P_{.j} = \sum_{j=1}^{n} P_{ij} = P_{1j} + P_{2j} + \dots + P_{nj}$

Note: The set $\{y_j, P_{.j}\}$ is called the marginal distribution of Y.

Conditional probability distribution:

$$P\{X=x_i / Y=y_j\} = \frac{P\{X=x_i \,\&\, Y=y_j\}}{P(Y=y_j)} = \frac{P_{ij}}{P_{.j}}$$ is called the conditional probability fun/. of X, given $Y=y_j$. The collection of pairs $\left\{x_i, \frac{P_{ij}}{P_{.j}}\right\}$, $i=1,2,\dots$ is called the conditional probability distribution of X, given $Y=y_j$. Similarly, the collection of pairs, $\left\{y_j, \frac{P_{ij}}{P_{i.}}\right\}$, $j=1,2,\dots$ is called the conditional probability distribution of Y given $X=x_i$.

Let (X,Y) be the two dimensional continuous r.v.. The conditional p.d.f. of X given Y is denoted by $f(x/y)$ & is defined as $f(x/y) = \frac{f(x,y)}{f(y)}$.

Similary, the conditional p.d.f. of Y given X is denoted by $f(y/x)$ & is defined as, $f(y/x) = \frac{f(x,y)}{f(x)}$.

Independent random variables:

Two RVs X & Y are said to be independent if $f(x,y) = f(x).f(y)$ where $f(x,y)$ is the joint p.d.f. of (X,Y), $f(x)$ is the marginal density func. of X & $f(y)$ is the marginal density func. of Y.

The r.vs X & Y are said to be independent r.vs if $P_{ij} = P_{i.} \times P_{.j}$ where P_{ij} is the joint probability func. of (X,Y), $P_{i.}$ is the marginal probability func. of X & $P_{.j}$ is the marginal probability func. of Y.

Joint probability density func.:

If (X,Y) is a two-dimensional continuous r.v. such that $P\{x - \frac{dx}{2} \le X \le x + \frac{dx}{2}, y - \frac{dy}{2} \le Y \le y + \frac{dy}{2}\} = f(x,y)\,dx\,dy$, then $f(x,y)$ is called the joint p.d.f. of (X,Y), provided $f(x,y)$ satisfies the following conditions.

(i) $f(x,y) \ge 0, \forall (x,y) \in R$, where R is the range space.

(ii) $\iint_R f(x,y)dx\,dy = 1$

In particular, $P(a \le X \le b, c \le Y \le d) = \int_c^d \int_a^b f(x,y)dx\,dy$

Cumulative distribution func.:

If (X,Y) is a two-dimensional continuous r.v., then $F(x,y) = P(X \le x \,\&\, Y \le y)$ is called the c.d.f. of (X,Y) & is defined as, $F(x,y) = \int_{-\infty}^{y}\int_{-\infty}^{x} f(x,y)dx\,dy$.

Marginal density func.:

If (X,Y) is a two-dimensional continuous r.v. such that

$$P\{x - \frac{dx}{2} \le X \le x + \frac{dx}{2}, -\infty < Y < \infty\} = \int_{-\infty}^{\infty}\int_{x-\frac{dx}{2}}^{x+\frac{dx}{2}} f(x,y)\,dy\,dx.$$

Let (X,Y) be the two dimensional r.v.. Then the marginal p.d.f. of X is denoted by $f(x)$ & is defined as, $f(x) = f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$.

Similarly the marginal p.d.f. of Y is denoted by $f(y)$ & is defined as,

$f(y) = f_Y(y) = \int_{-\infty}^{\infty} f(x,y)dx$.

Joint probability density funⁿ:

Let (X, Y) be the two dimensional r.v. & $F(x,y)$ be the joint probability distribution funⁿ. Then the joint p.d.f. of X & Y is denoted by $f(x,y)$ & is defined as, $f(x,y) = \frac{\partial^2 F(x,y)}{\partial x\, \partial y}$.

Problems:

① The joint probability mass funⁿ of (X,Y) is given by $P(x,y) = K(2x+3y)$, $x = 0,1,2$; $y = 1,2,3$. Find all the marginal & conditional probability distributions. Also find the probability distribution of $(X+Y)$ & $P(X+Y>3)$.

Sol:

X \ Y	1	2	3
0	3K	6K	9K
1	5K	8K	11K
2	7K	10K	13K

$\sum_{j=1}^{3} \sum_{i=1}^{3} P(x_i, y_j) = 1$

$\Rightarrow 72K = 1 \Rightarrow K = 1/72$

X \ Y	1	2	3	$P_X(x) = f(x) = P_{i\cdot}$
0	$3/72$	$6/72$	$9/72$	$P(X=0) = 18/72$
1	$5/72$	$8/72$	$11/72$	$P(X=1) = 24/72$
2	$7/72$	$10/72$	$13/72$	$P(X=2) = 30/72$
$P_Y(y) = f(y) = P_{\cdot j}$	$P(Y=1) = \frac{15}{72}$	$P(Y=2) = \frac{24}{72}$	$P(Y=3) = \frac{33}{72}$	

Marginal distributions of X:

$P(X=0) = \frac{18}{72}$; $P(X=1) = \frac{24}{72}$; $P(X=2) = \frac{30}{72}$

Marginal distributions of Y:

$P(Y=1) = \frac{15}{72}$; $P(Y=2) = \frac{24}{72}$; $P(Y=3) = \frac{33}{72}$

Conditional distribution of X, given Y is $P\{X = x_i / Y = y_j\}$

$P(X=0/Y=1) = \frac{P(X=0, Y=1)}{P(Y=1)} = \frac{3/72}{15/72} = \frac{1}{5}$

$P(X=1/Y=1) = \frac{P(X=1, Y=1)}{P(Y=1)} = \frac{5/72}{15/72} = \frac{1}{3}$

$P(X=2/Y=1) = \frac{P(X=2, Y=1)}{P(Y=1)} = \frac{7/72}{15/72} = \frac{7}{15}$

$P(X=0/Y=2) = \frac{P(X=0, Y=2)}{P(Y=2)} = \frac{6/72}{24/72} = \frac{1}{4}$

$P(X=1/Y=2) = \frac{P(X=1, Y=2)}{P(Y=2)} = \frac{8/72}{24/72} = \frac{1}{3}$

$P(X=2/Y=2) = \frac{P(X=2, Y=2)}{P(Y=2)} = \frac{10/72}{24/72} = \frac{5}{12}$

$P(X=0/Y=3) = \frac{P(X=0, Y=3)}{P(Y=3)} = \frac{9/72}{33/72} = \frac{9}{33} = \frac{3}{11}$

$P(X=1/Y=3) = \frac{P(X=1, Y=3)}{P(Y=3)} = \frac{11/72}{33/72} = \frac{1}{3}$

$P(X=2/Y=3) = \frac{P(X=2, Y=3)}{P(Y=3)} = \frac{13/72}{33/72} = \frac{13}{33}$

Conditional distribution of Y, given X is $P\{Y=y_j \mid X=x_i\}$

$$P(Y=1/X=0)=\frac{P(X=0,Y=1)}{P(X=0)}=\frac{3/72}{18/72}=\frac{1}{6}$$

$$P(Y=2/X=0)=\frac{P(X=0,Y=2)}{P(X=0)}=\frac{6/72}{18/72}=\frac{1}{3}$$

$$P(Y=3/X=0)=\frac{P(X=0,Y=3)}{P(X=0)}=\frac{9/72}{18/72}=\frac{1}{2}$$

$$P(Y=1/X=1)=\frac{P(X=1,Y=1)}{P(X=1)}=\frac{5/72}{24/72}=\frac{5}{24}$$

$$P(Y=2/X=1)=\frac{P(X=1,Y=2)}{P(X=1)}=\frac{8/72}{24/72}=\frac{1}{3}$$

$$P(Y=3/X=1)=\frac{P(X=1,Y=3)}{P(X=1)}=\frac{11/72}{24/72}=\frac{11}{24}$$

$$P(Y=1/X=2)=\frac{P(X=2,Y=1)}{P(X=2)}=\frac{7/72}{30/72}=\frac{7}{30}$$

$$P(Y=2/X=2)=\frac{P(X=2,Y=2)}{P(X=2)}=\frac{10/72}{30/72}=\frac{1}{3}$$

$$P(Y=3/X=2)=\frac{P(X=2,Y=3)}{P(X=2)}=\frac{13/72}{30/72}=\frac{13}{30}$$

Probability distribution of X+Y :

$X+\cancel{y}Y$	Probability
1	$P(0,1)=3/72$
2	$P(0,2)+P(1,1)=\frac{6}{72}+\frac{5}{72}=\frac{11}{72}$
3	$P(0,3)+P(1,2)+P(2,1)=\frac{9}{72}+\frac{8}{72}+\frac{7}{72}=\frac{24}{72}$
4	$P(1,3)+P(2,2)=\frac{11}{72}+\frac{10}{72}=\frac{21}{72}$
5	$P(2,3)=\frac{13}{72}$

$$P[X+Y>3]=P[X+Y=4]+P[X+Y=5]=\frac{21}{72}+\frac{13}{72}=\frac{34}{72}$$

② The joint probability mass fun/. (p.m.f.) of X & Y is

X \ Y	0	1	2
0	0.1	0.04	0.02
1	0.08	0.2	0.06
2	0.06	0.14	0.3

Compute the marginal p.m.f. of X & Y, $P[X\le 1, Y\le 1]$ & check if X & Y are independent.

Sol:

X \ Y	0	1	2	$P(X=x_i)=P_{i.}$
0	0.1	0.04	0.02	$P(X=0)=0.16$
1	0.08	0.2	0.06	$P(X=1)=0.34$
2	0.06	0.14	0.3	$P(X=2)=0.5$
$P(Y=y_j)$ $=P_{.j}$	$P(Y=0)$ $=0.24$	$P(Y=1)$ $=0.38$	$P(Y=2)$ $=0.38$	

The marginal p.m.f. of X are $P(X=0)=0.16$; $P(X=1)=0.34$ & $P(X=2)=0.5$

The marginal p.m.f. of Y are $P(Y=0)=0.24$; $P(Y=1)=0.38$ & $P(Y=2)=0.38$

Now, $P[X\le 1, Y\le 1] = P[X=0,Y=0]+P[X=0,Y=1]+P[X=1,Y=0]+P[X=1,Y=1]$

$$=0.1+0.04+0.08+0.2=0.42$$

If $P_{ij}=P_{i.}\times P_{.j}$ then we can say that X & Y are independent.

We have $P_{0.}=0.16$ & $P_{.0}=0.24$

$\therefore P_{0.}\times P_{.0}=0.0384\neq 0.1=P_{00}$

$\therefore P_{ij}\neq P_{i.}\times P_{.j}$

Hence X & Y are not independent.

③ Suppose the joint p.d.f is given by $f(x,y)=\begin{cases}\frac{6}{5}(x+y^2) & ; 0\le x\le 1, 0\le y\le 1\\ 0, & \text{otherwise}\end{cases}$. Obtain the marginal p.d.f. of X & that of Y. Hence or otherwise find $P[\frac{1}{4}\le y\le \frac{3}{4}]$.

Sol: Given that $f(x,y)=\begin{cases}\frac{6}{5}(x+y^2), & 0\le x\le 1, 0\le y\le 1\\ 0, & \text{otherwise}\end{cases}$

The marginal p.d.f. of X is $f(x)=\int_{-\infty}^{\infty} f(x,y)\,dy = \frac{6}{5}\int_0^1 (x+y^2)\,dy$

$$=\frac{6}{5}\left[xy+\frac{y^3}{3}\right]_0^1=\frac{6}{5}\left[x+\frac{1}{3}\right], \quad 0\le x\le 1$$

The marginal p.d.f. of Y is $f(y)=\int_{-\infty}^{\infty} f(x,y)\,dx=\frac{6}{5}\int_0^1 (x+y^2)\,dx$

$$=\frac{6}{5}\left[\frac{x^2}{2}+y^2x\right]_0^1=\frac{6}{5}\left[\frac{1}{2}+y^2\right], \quad 0\le y\le 1$$

$$P\left[\frac{1}{4}\le y\le\frac{3}{4}\right]=\int_{1/4}^{3/4} f(y)\,dy=\int_{1/4}^{3/4}\frac{6}{5}\left(\frac{1}{2}+y^2\right)dy=\frac{6}{5}\left[\frac{1}{2}y+\frac{y^3}{3}\right]_{1/4}^{3/4}$$

$$=\frac{6}{5}\left[\frac{3}{8}+\frac{9}{64}-\frac{1}{8}-\frac{1}{192}\right]=\frac{6}{5}\left[\frac{1}{4}+\frac{26}{192}\right]=\frac{6}{5}\left[\frac{48+26}{192}\right]$$

$$=\frac{6}{5}\times\frac{74}{192}=0.4625$$

④ Let X & Y have joint p.d.f. $f(x,y)=2$, $0<x<y<1$. Find the m.d.f. find the conditional density func. of Y given X=x.

Sol: The marginal density func. of X is given by

$$f_X(x)=f(x)=\int_{-\infty}^{\infty} f(x,y)\,dy=\int_x^1 2\,dy=2(y)_x^1=2(1-x), \quad 0<x<1$$

The marginal density func. of Y is given by

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$$f_Y(y) = f(y) = \int_{-\infty}^{\infty} f(x,y)dx = \int_0^y 2dx = 2(x)_0^y = 2y,\ 0<y<1$$

The conditional density funl. of Y given X=x is,

$$f(Y/X) = \frac{f(x,y)}{f(x)} = \frac{2}{2(1-x)} = \frac{1}{1-x}$$

---

⑤ If the joint p.d.f. of a 2 dimensional r.v. (X,Y) is given by $f(x,y)=\begin{cases} 2, & 0<y<x<1 \\ 0, & \text{otherwise} \end{cases}$ Find the marginal density funl. of X & Y. Also find X & Y are independent.

Sol: The marginal density funl. of X is given by

$$f(x) = \int_{-\infty}^{\infty} f(x,y)dy = \int_0^x 2dy = 2(y)_0^x = 2x,\ 0<x<1$$

The marginal density funl. of Y is given by

$$f(y) = \int_{-\infty}^{\infty} f(x,y)dx = \int_y^1 2dx = 2(x)_y^1 = 2(1-y),\ 0<y<1$$

$$f(x).f(y) = (2x)(2(1-y)) = 4x(1-y) \neq 2 = f(x,y)$$

$\therefore f(x,y) \neq f(x).f(y)$. Hence the rvs X & Y are dependent on each other.

---

⑥ The joint p.d.f. of the r.v. (X,Y) is given by $f(x,y) = kxye^{-(x^2+y^2)}, x>0, y>0$. Find the value of k & prove also that X & Y are independent.

Sol: WKT $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)dxdy = 1 \Rightarrow \int_0^{\infty}\int_0^{\infty} kxye^{-(x^2+y^2)}dxdy = 1$

$$\Rightarrow k\int_0^{\infty}\int_0^{\infty} xye^{-x^2}e^{-y^2}dxdy = 1$$

Put $x^2=t$
$2xdx = dt \Rightarrow xdx = \frac{dt}{2}$
when $x=0, t=0$
$x \to \infty, t \to \infty$

$$\Rightarrow k\int_0^{\infty}\int_0^{\infty} e^{-t}\frac{dt}{2}\, ye^{-y^2}dy = 1$$

$$\Rightarrow \frac{k}{2}\int_0^{\infty}\left(\frac{e^{-t}}{-1}\right)_0^{\infty} ye^{-y^2}dy = 1 \Rightarrow \frac{-k}{2}\int_0^{\infty}(0-1)ye^{-y^2}dy = 1$$

Put $y^2 = u$
$2ydy = du \Rightarrow ydy = \frac{du}{2}$
when $y=0, u=0$
$y \to \infty, u \to \infty$

$$\Rightarrow \frac{k}{2}\int_0^{\infty} ye^{-y^2}dy = 1 \Rightarrow \frac{k}{2}\int_0^{\infty} e^{-u}\frac{du}{2} = 1$$

$$\Rightarrow \frac{k}{4}\left(\frac{e^{-u}}{-1}\right)_0^{\infty} = 1 \Rightarrow -\frac{k}{4}(0-1) = 1 \Rightarrow k/4 = 1 \Rightarrow k = 4$$

The marginal density funl. of X is given by

$$f(x) = \int_{-\infty}^{\infty} f(x,y)dy = \int_0^{\infty} 4xye^{-(x^2+y^2)}dy = 4xe^{-x^2}\int_0^{\infty} ye^{-y^2}dy$$

$$= 4xe^{-x^2}\int_0^{\infty} e^{-u}\frac{du}{2} = 2xe^{-x^2}\left(\frac{e^{-u}}{-1}\right)_0^{\infty} = 2xe^{-x^2},\ x>0$$

The marginal density fun/. of Y is given by

$$f(y)=\int_{-\infty}^{\infty} f(x,y)\,dx = \int_{0}^{\infty} 4xye^{-(x^2+y^2)}\,dx = 4ye^{-y^2}\int_{0}^{\infty} xe^{-x^2}\,dx$$

$$= 4ye^{-y^2}\int_{0}^{\infty} e^{-t}\,\frac{dt}{2} = 2ye^{-y^2}\left(\frac{e^{-t}}{-1}\right)_{0}^{\infty} = 2ye^{-y^2}, \; y>0$$

Now, $f(x)\,f(y) = 2xe^{-x^2}\cdot 2ye^{-y^2} = 4xye^{-(x^2+y^2)} = f(x,y)$

∴ X & Y are independent.

⑦ Given $f_{xy}(x,y)=\begin{cases} cx(x-y), & 0<x<2, \; -x<y<x \\ 0, & \text{elsewhere} \end{cases}$ (a) Evaluate c (b) Find $f_x(x)$

(c) $f_{Y/X}(y/x)$ & (d) $= f_Y(y)$.

Sol: (a) WKT $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1 \Rightarrow \int_{0}^{2}\int_{-x}^{x} cx(x-y)\,dy\,dx = 1$

$$\Rightarrow c\int_{0}^{2} x\left(xy-\frac{y^2}{2}\right)_{-x}^{x} dx = 1 \Rightarrow c\int_{0}^{2} x\left(x^2-\frac{x^2}{2}+x^2+\frac{x^2}{2}\right)dx = 1$$

$$\Rightarrow c\int_{0}^{2} 2x^3\,dx = 1 \Rightarrow 2c\left(\frac{x^4}{4}\right)_{0}^{2} = 1 \Rightarrow \frac{c}{2}(16) = 1 \Rightarrow c = 1/8$$

(b) Marginal density fun/. of X is given by

$$f_x(x) = f(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \frac{1}{8}\int_{-x}^{x} x(x-y)\,dy = \frac{x}{8}\left(xy-\frac{y^2}{2}\right)_{-x}^{x}$$

$$= \frac{x}{8}\left(x^2-\frac{x^2}{2}+x^2+\frac{x^2}{2}\right) = \frac{x^3}{4}, \; 0<x<2$$

(c) $f_{Y/X}(y/x) = \dfrac{f(x,y)}{f(x)} = \dfrac{\frac{1}{8}x(x-y)}{x^3/4} = \dfrac{1}{2x^2}(x-y), \; -x<y<x$

(d) Marginal density fun/. of Y is given by

$$f_Y(y) = f(y) = \int_{-\infty}^{\infty} f(x,y)\,dx = \frac{1}{8}\int_{0}^{2} x(x-y)\,dx = \frac{1}{8}\int_{0}^{2}(x^2-xy)\,dx$$

$$= \frac{1}{8}\left(\frac{x^3}{3}-\frac{x^2y}{2}\right)_{0}^{2} = \frac{1}{8}\left(\frac{8}{3}-2y\right) = \frac{1}{24}(8-6y) = \frac{1}{12}(4-3y), \; -x<y<x$$

⑧ The joint p.d.f. of (X,Y) is given by $f(x,y)=e^{-(x+y)}, \; 0\le x,y<\infty$. Are X & Y independent? Why?

Sol: The marginal density fun/. of X is given by

$$f(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \int_{0}^{\infty} e^{-(x+y)}\,dy = e^{-x}\int_{0}^{\infty} e^{-y}\,dy = e^{-x}\left(\frac{e^{-y}}{-1}\right)_{0}^{\infty} = e^{-x}, \; 0\le x<\infty$$

Marginal density fun/. of Y is given by

$$f(y) = \int_{-\infty}^{\infty} f(x,y)\,dx = \int_{0}^{\infty} e^{-(x+y)}\,dx = e^{-y}\left(\frac{e^{-x}}{-1}\right)_{0}^{\infty} = e^{-y}, \; 0\le y<\infty$$

Consider, $f(x)\cdot f(y) = e^{-x}\cdot e^{-y} = e^{-(x+y)} = f(x,y)$. Hence X & Y are independent.

⑨ If the joint p.d.f. of a two-dimensional r.v. (X,Y) is given by
$f(x,y)=\begin{cases} x^2+\frac{xy}{3}, & 0<x<1\ ;\ 0<y<2 \\ 0, & \text{elsewhere}\end{cases}$. Find (i) $P(X>\frac{1}{2})$ (ii) $P(Y<X)$ & (iii) $P[Y<\frac{1}{2}/X<\frac{1}{2}]$

Check whether the conditional density funs/. are valid.

Sol: Marginal density fun/. of X is given by

$$f(x)=\int_{-\infty}^{\infty} f(x,y)\,dy=\int_0^2\left(x^2+\frac{xy}{3}\right)dy=\left(x^2y+\frac{xy^2}{6}\right)_0^2=2x^2+\frac{2x}{3}\ ,\ 0<x<1$$

Marginal density fun/. of Y is given by

$$f(y)=\int_{-\infty}^{\infty} f(x,y)\,dx=\int_0^1\left(x^2+\frac{xy}{3}\right)dx=\left(\frac{x^3}{3}+\frac{x^2y}{6}\right)_0^1=\frac{1}{3}+\frac{y}{6}\ ,\ 0<y<2$$

$$\text{(i) } P(X>\tfrac{1}{2})=\int_{1/2}^1 f(x)\,dx=\int_{1/2}^1\left(2x^2+\frac{2x}{3}\right)dx=\left(\frac{2x^3}{3}+\frac{x^2}{3}\right)_{1/2}^1$$

$$=\frac{2}{3}+\frac{1}{3}-\frac{1}{12}-\frac{1}{12}\neq\cancel{\frac{16+8-2-1}{24}}\neq\cancel{\frac{21}{24}}=\frac{8+4-1-1}{12}=\frac{10}{12}=\frac{5}{6}$$

$$\text{(ii) } P(Y<X)=\int_0^1\int_0^x\left(x^2+\frac{xy}{3}\right)dy\,dx=\int_0^1\left(x^2y+\frac{xy^2}{6}\right)_0^x dx=\int_0^1\left(x^3+\frac{x^3}{6}\right)dx$$

$$=\left(\frac{x^4}{4}+\frac{x^4}{24}\right)_0^1=\frac{1}{4}+\frac{1}{24}=\frac{6+1}{24}=\frac{7}{24}$$

$$\text{(iii) } P[Y<\tfrac{1}{2}/X<\tfrac{1}{2}]=\frac{P[X<\frac{1}{2},Y<\frac{1}{2}]}{P[X<\frac{1}{2}]}$$

$$P[X<\tfrac{1}{2},Y<\tfrac{1}{2}]=\int_0^{1/2}\int_0^{1/2}\left(x^2+\frac{xy}{3}\right)dx\,dy=\int_0^{1/2}\left(\frac{x^3}{3}+\frac{x^2y}{6}\right)_0^{1/2}dy$$

$$=\int_0^{1/2}\left(\frac{1}{24}+\frac{y}{24}\right)dy=\frac{1}{24}\int_0^{1/2}(1+y)\,dy=\frac{1}{24}\left(y+\frac{y^2}{2}\right)_0^{1/2}$$

$$=\frac{1}{24}\left(\frac{1}{2}+\frac{1}{8}\right)=\frac{1}{24}\left(\frac{4+1}{8}\right)=\frac{5}{192}$$

$$P(X<\tfrac{1}{2})=\int_0^{1/2} f(x)\,dx=\int_0^{1/2}\left(2x^2+\frac{2x}{3}\right)dx=\left(\frac{2x^3}{3}+\frac{x^2}{3}\right)_0^{1/2}=\frac{1}{12}+\frac{1}{12}=\frac{1}{6}$$

$$\therefore P[Y<\tfrac{1}{2}/X<\tfrac{1}{2}]=\frac{5/192}{1/6}=\frac{5}{192}\times 6=\frac{5}{32}$$

Checking the conditional density funs/. are valid.

$$\int_0^1 f(x/y)\,dx=\int_0^1\frac{f(x,y)}{f(y)}\,dx=\int_0^1\left(\frac{x^2+xy/3}{1/3+y/6}\right)dx=\int_0^1\left(\frac{3x^2+xy}{3}\times\frac{6}{2+y}\right)dx$$

$$=\int_0^1\left(\frac{6x^2+2xy}{2+y}\right)dx=\frac{2}{2+y}\int_0^1(3x^2+xy)\,dx=\frac{2}{2+y}\left(x^3+\frac{x^2y}{2}\right)_0^1$$

$$=\frac{2}{2+y}\left(1+\frac{y}{2}\right)=\frac{2}{2+y}\left(\frac{2+y}{2}\right)=1$$

& $\int_0^2 f(y/x)\,dy = \int_0^2 \frac{f(x,y)}{f(x)}\,dy = \int_0^2 \left(\frac{3x^2+xy}{3} \times \frac{3}{6x^2+2x}\right)dy$

$= \frac{1}{6x^2+2x}\int_0^2 (3x^2+xy)\,dy = \frac{1}{6x^2+2x}\left(3x^2y+\frac{xy^2}{2}\right)_0^2$

$= \frac{1}{6x^2+2x}\left(6x^2+2x\right) = 1$

⑩ If the joint p.d.f. of a two-dimensional r.v. (X,Y) is given by $f(x,y) = \begin{cases} K(6-x-y), & 0<x<2,\ 2<y<4 \\ 0, & \text{otherwise} \end{cases}$. Find (i) the value of K (ii) $P(X<1, Y<3)$ (iii) $P(X+Y<3)$ & (iv) $P(X<1 / Y<3)$.

Sol: (i) WKT $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1 \Rightarrow \int_2^4\int_0^2 K(6-x-y)\,dx\,dy = 1$

$\Rightarrow K\int_2^4 \left(6x-\frac{x^2}{2}-xy\right)_0^2 dy = 1 \Rightarrow K\int_2^4 (12-2-2y)\,dy = 1$

$\Rightarrow K\int_2^4 (10-2y)\,dy = 1 \Rightarrow 2K\int_2^4 (5-y)\,dy = 1 \Rightarrow 2K\left(5y-\frac{y^2}{2}\right)_2^4 = 1$

$\Rightarrow 2K(20-8-10+2) = 1 \Rightarrow 8K = 1 \Rightarrow K = 1/8$

(ii) $P(X<1, Y<3) = \int_0^1\int_2^3 f(x,y)\,dy\,dx = \frac{1}{8}\int_0^1\int_2^3 (6-x-y)\,dy\,dx$

$= \frac{1}{8}\int_0^1 \left(6y-xy-\frac{y^2}{2}\right)_2^3 dx = \frac{1}{8}\int_0^1 \left(18-3x-\frac{9}{2}-12+2x+2\right)dx$

$= \frac{1}{8}\int_0^1 \left(\frac{7}{2}-x\right)dx = \frac{1}{8}\left(\frac{7x}{2}-\frac{x^2}{2}\right)_0^1 = \frac{1}{16}(7-1) = \frac{3}{8}$

(iii) $P(X+Y<3) = \int_2^3\int_0^{3-y} f(x,y)\,dx\,dy = \frac{1}{8}\int_2^3\int_0^{3-y} (6-x-y)\,dx\,dy$

$P(X+Y<3) = \int_0^1\int_2^{3-x} f(x,y)\,dy\,dx$

Min. value for y is 2 $x+y<3$ means x value is 1.

$= \frac{1}{8}\int_2^3 \left(6x-\frac{x^2}{2}-xy\right)_0^{3-y} dy = \frac{1}{8}\int_2^3 \left(18-6y-\frac{(3-y)^2}{2}-(3-y)y\right)dy$

$= \frac{1}{8}\int_2^3 \left(18-6y-\frac{1}{2}(3-y)^2-3y+y^2\right)dy = \frac{1}{8}\int_2^3 \left(18-9y+y^2-\frac{1}{2}(3-y)^2\right)dy$

$= \frac{1}{8}\left[18y-\frac{9y^2}{2}+\frac{y^3}{3}-\frac{1}{2}\frac{(3-y)^3}{3(-1)}\right]_2^3$

$= \frac{1}{8}\left[54-\frac{81}{2}+9+0-36+18-\frac{8}{3}-\frac{1}{6}\right] = \frac{1}{8}\left[45+\frac{-243-16-1}{6}\right]$

$= \frac{1}{8}\left[45-\frac{130}{3}\right] = \frac{1}{8}\times\frac{5}{3} = \frac{5}{24}$

(iv) $P(X<1 / Y<3) = \frac{P(X<1, Y<3)}{P(Y<3)}$

$$P(Y<3) = \int_2^3 f(y)\,dy$$

$$f(y) = \int_{-\infty}^{\infty} f(x,y)\,dx = \int_0^2 \frac{1}{8}(6-x-y)\,dx = \frac{1}{8}\left(6x - \frac{x^2}{2} - xy\right)_0^2$$

$$= \frac{1}{8}(12-2-2y) = \frac{1}{8}(10-2y) = \frac{1}{4}(5-y), \quad 2<y<4$$

$$\therefore P(Y<3) = \int_2^3 \frac{1}{4}(5-y)\,dy = \frac{1}{4}\left(5y - \frac{y^2}{2}\right)_2^3 = \frac{1}{4}\left(15 - \frac{9}{2} - 10 + 2\right) = \frac{1}{4} \times \frac{5}{2} = \frac{5}{8}$$

$$\therefore P(X<1/Y<3) = \frac{3/8}{5/8} = \frac{3}{5}$$

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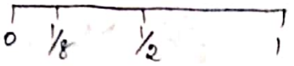
⑪ The joint density funct. of the rvs X & Y is given by $f(x,y) = \begin{cases} 8xy, & 0<x<1;\ 0<y<x \\ 0, & \text{elsewhere} \end{cases}$. Find $P(Y<1/8 \,/\, X<1/2)$. Also find the conditional density funct. $f(y/x)$.

Sol: $$P(Y<1/8 \,/\, X<1/2) = \frac{P(X<1/2,\ Y<1/8)}{P(X<1/2)}$$

$$P(X<1/2,\ Y<1/8) = \int_0^{1/2}\int_0^{x} f(x,y)\,dy\,dx = \int_0^{1/2}\int_0^{x} 8xy\,dy\,dx = 8\int_0^{1/2} x\left(\frac{y^2}{2}\right)_0^x dx$$

$$= \frac{8}{2}\int_0^{1/2} x(x^2)\,dx = 4\int_0^{1/2} x^3\,dx = 4\left(\frac{x^4}{4}\right)_0^{1/2} = \frac{1}{16}$$



$$P(X<1/2) = \int_0^{1/2} f(x)\,dx$$

$$f(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \int_0^x 8xy\,dy = 8x\left(\frac{y^2}{2}\right)_0^x = 4x(x^2) = 4x^3, \quad 0<x<1$$

$$\therefore P(X<1/2) = \int_0^{1/2} 4x^3\,dx = 4\left(\frac{x^4}{4}\right)_0^{1/2} = \frac{1}{16}$$

$$\therefore P(Y<1/8 \,/\, X<1/2) = \frac{1/16}{1/16} = 1$$

$$f(y/x) = \frac{f(x,y)}{f(x)} = \frac{8xy}{4x^3} = \frac{2y}{x^2}, \quad 0<x<1$$

---

⑫ If the joint density funct. of the two rvs X & Y be $f(x,y) = \begin{cases} e^{-(x+y)}, & x\geq 0, y\geq 0 \\ 0, & \text{otherwise} \end{cases}$

Find (i) $P(X<1)$ & (ii) $P(X+Y<1)$.

Sol: Marginal density funct. of X is given by

$$f(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \int_0^{\infty} e^{-x}.e^{-y}\,dy = e^{-x}\left(\frac{e^{-y}}{-1}\right)_0^{\infty} = e^{-x}, \quad x\geq 0$$

$$\text{(i)}\ P(X<1) = \int_0^1 f(x)\,dx = \int_0^1 e^{-x}\,dx = \left(\frac{e^{-x}}{-1}\right)_0^1 = -(e^{-1}-1) = 1-e^{-1}$$

$$\text{(ii)}\ P(X+Y<1) = \int_0^1\int_0^{1-y} f(x,y)\,dx\,dy = \int_0^1\int_0^{1-y} e^{-x}.e^{-y}\,dx\,dy = \int_0^1 e^{-y}\left(\frac{e^{-x}}{-1}\right)_0^{1-y} dy$$

$$= -\int_0^1 e^{-y}\left(e^{-(1-y)} - 1\right) dy = \int_0^1 e^{-y}\left(1 - e^{-1+y}\right) dy$$

$$= \int_0^1 \left(e^{-y} - e^{-1}\right) dy = \left[\frac{e^{-y}}{-1} - e^{-1}y\right]_0^1 = \left[-e^{-y} - e^{-1}y\right]_0^1$$

$$= \left[-e^{-1} - e^{-1} + 1\right] = 1 - 2e^{-1}$$

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⑬ X & Y are two rvs having joint density fun/. $f(x,y) = \begin{cases} \frac{1}{8}(6-x-y), & 0<x<2, \; 2<y<4 \\ 0, & \text{otherwise} \end{cases}$

Find $P(X<1 \cap Y<3)$.

Sol: $P(X<1 \cap Y<3) = \int_0^1 \int_2^3 \frac{1}{8}(6-x-y)\,dy\,dx$

$$= \frac{1}{8}\int_0^1 \left(6y - xy - \frac{y^2}{2}\right)_2^3 dx = \frac{1}{8}\int_0^1 \left(18 - 3x - \frac{9}{2} - 12 + 2x + 2\right) dx$$

$$= \frac{1}{8}\int_0^1 \left(\frac{7}{2} - x\right) dx = \frac{1}{8}\left(\frac{7x}{2} - \frac{x^2}{2}\right)_0^1 = \frac{1}{16}(7-1) = \frac{3}{8}$$

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⑭ Given the joint p.d.f. of (X,Y) as $f(x,y) = \begin{cases} 8xy, & 0<x<y<1 \\ 0, & \text{otherwise} \end{cases}$. Find the marginal & conditional pdfs of X & Y. Are X & Y independent?

Sol: Marginal density fun/. of X is given by

$$f(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \int_x^1 8xy\,dy = 8x\left(\frac{y^2}{2}\right)_x^1 = 4x(1-x^2), \quad 0<x<1$$

Marginal density fun/. of Y is given by

$$f(y) = \int_{-\infty}^{\infty} f(x,y)\,dx = \int_0^y 8xy\,dx = 8y\left(\frac{x^2}{2}\right)_0^y = 4y(y^2) = 4y^3, \quad 0<y<1$$

$$f(x/y) = \frac{f(x,y)}{f(y)} = \frac{8xy}{4y^3} = \frac{2x}{y^2}, \quad 0<x<y$$

$$f(y/x) = \frac{f(x,y)}{f(x)} = \frac{8xy}{4x(1-x^2)} = \frac{2y}{1-x^2}, \quad x<y<1$$

$$f(x).f(y) = 4x(1-x^2).4y^3 = 16xy^3(1-x^2) \neq f(x,y)$$

Hence X & Y are not independent.

Covariance:

(i) $\mathrm{Cov}(X,Y) = E(XY) - E(X)E(Y)$

(ii) $\mathrm{Cov}(aX, bY) = ab\,\mathrm{cov}(X,Y)$

(iii) $\mathrm{Cov}(X+a, Y+b) = \mathrm{Cov}(X,Y)$

(iv) $\mathrm{Cov}(aX+b, cY+d) = ac\,\mathrm{Cov}(X,Y)$

(v) $V(X_1+X_2) = V(X_1)+V(X_2)+2\,\mathrm{Cov}(X_1,X_2)$

(vi) $V(X_1-X_2) = V(X_1)+V(X_2) - 2\,\mathrm{Cov}(X_1,X_2)$

Note: If X & Y are independent, then $\mathrm{Cov}(X,Y) = 0$

($\because$ X & Y are independent, $E(XY) = E(X).E(Y)$)

Karl Pearson's coefficient of correlation:

Let X & Y be given random variables. The Karl Pearson's coefficient of correlation is denoted by r_{XY} or $r(X,Y)$ & defined as

$$r(X,Y) = r_{XY} = \frac{Cov(X,Y)}{\sqrt{Var(X)}\,\sqrt{Var(Y)}} = \frac{Cov(X,Y)}{\sigma_X \cdot \sigma_Y}$$

where $Cov(X,Y) = E(XY) - E(X)E(Y) = \frac{\sum xy}{n} - \bar{X}\bar{Y}$. Here $\bar{X} = \frac{\sum x}{n}$ & $\bar{Y} = \frac{\sum y}{n}$ & n is the no. of items in the given data.

$\sigma_X^2 = Var(X) = \frac{1}{n}\sum X^2 - \bar{X}^2$ & $\sigma_Y^2 = Var(Y) = \frac{1}{n}\sum Y^2 - \bar{Y}^2$

Note: (i) Correlation coefficient always lies between -1 to 1.

(ii) Two rvs with non zero correlation are said to be correlated.

Rank Correlation:

If (x_i, y_i), $i = 1, 2, \ldots, n$ be the ranks of the individuals in two characteristics A & B respectively. Then the rank correlation coefficient is given by

$$r = 1 - \frac{6}{n(n^2-1)} \sum_{i=1}^{n} d_i^2 \quad \text{where } d_i = x_i - y_i \;;\; n = \text{no. of items.}$$

where d_i is the different between the ranks. This formula is called Spearman's formula for the rank correlation coefficient.

Note: In the correction formula, we add the factor $\frac{m(m^2-1)}{12}$ to $\sum d^2$ where m is the no. of items an item is repeated. This correction factor is to be added for each repeated value.

Problems:

① Calculate the correlation coefficient for the following heights (in inches) of fathers X their sons Y.

X : 65 66 67 67 68 69 70 72

Y : 67 68 65 68 72 72 69 71

Sol:

X	Y	XY	X^2	Y^2
65	67	4355	4225	4489
66	68	4488	4356	4624
67	65	4355	4489	4225
67	68	4556	4489	4624
68	72	4896	4624	5184
69	72	4968	4761	5184
70	69	4830	4900	4761
72	71	5112	5184	5041

Here $\sum X = 544$, $\sum Y = 552$

$\sum XY = 37560$

$\sum X^2 = 37028$

$\sum Y^2 = 38132$

$\bar{X} = \frac{\Sigma x}{n} = \frac{544}{8} = 68$; $\bar{Y} = \frac{\Sigma y}{n} = \frac{552}{8} = 69$

$\bar{X}\bar{Y} = 68 \times 69 = 4692$

$\sigma_x = \sqrt{\frac{1}{n}\Sigma x^2 - \bar{X}^2} = \sqrt{\frac{1}{8}(37028) - (68)^2} = \sqrt{4628.5 - 4624} = 2.1213$

$\sigma_y = \sqrt{\frac{1}{n}\Sigma y^2 - \bar{Y}^2} = \sqrt{\frac{1}{8}(38132) - (69)^2} = \sqrt{4766.5 - 4761} = 2.3452$

$Cov(x,y) = \frac{1}{n}\Sigma xy - \bar{X}\bar{Y} = \frac{1}{8}(37560) - (68 \times 69) = 4695 - 4692 = 3$

The correlation coefficient of X & Y is given by

$r(x,y) = \frac{Cov(x,y)}{\sigma_x \cdot \sigma_y} = \frac{3}{(2.1213)(2.3452)} = \frac{3}{4.9749} = 0.603$

② Two rvs X&Y have the joint density $f(x,y) = \begin{cases} 2-x-y, & 0<x<1, 0<y<1 \\ 0, & \text{otherwise} \end{cases}$. Show that $Cor(x,y) = \frac{-1}{11}$.

Sol: Marginal density funⁿ. of X is

$f(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \int_0^1 (2-x-y)\,dy = \left[2y - xy - \frac{y^2}{2}\right]_0^1 = 2 - x - \frac{1}{2} = \frac{3}{2} - x \;, 0<x<1$

Marginal density funⁿ. of Y is

$f(y) = \int_{-\infty}^{\infty} f(x,y)\,dx = \int_0^1 (2-x-y)\,dx = \left(2x - \frac{x^2}{2} - xy\right)_0^1 = 2 - \frac{1}{2} - y = \frac{3}{2} - y \;, 0<y<1$

Now, $E(x) = \int_{-\infty}^{\infty} x f(x)\,dx = \int_0^1 x\left(\frac{3}{2} - x\right)dx = \int_0^1 \left(\frac{3}{2}x - x^2\right)dx$

$= \left[\frac{3x^2}{4} - \frac{x^3}{3}\right]_0^1 = \frac{3}{4} - \frac{1}{3} = \frac{9-4}{12} = \frac{5}{12}$

Similarly, $E(Y) = \frac{5}{12}$

~~$E(x^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx = \int_0^1 x^2\left(\frac{3}{2} - x - y\right)dx = \int_0^1 (2x^2 - x^3 - x^2 y)\,dx$~~

~~$= \left[\frac{2x^3}{3} - \frac{x^4}{4} - \frac{x^3 y}{3}\right]_0^1 = \frac{2}{3} - \frac{1}{4} - \frac{y}{3}$~~

$E(x^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx = \int_0^1 x^2\left(\frac{3}{2} - x\right)dx = \int_0^1 \left(\frac{3x^2}{2} - x^3\right)dx$

$= \left[\frac{3x^3}{6} - \frac{x^4}{4}\right]_0^1 = \frac{1}{2} - \frac{1}{4} = \frac{2-1}{4} = \frac{1}{4}$

Similarly, $E(y^2) = \frac{1}{4}$

$\therefore \sigma_x^2 = Var(x) = E(x^2) - [E(x)]^2 = \frac{1}{4} - \left(\frac{5}{12}\right)^2 = \frac{1}{4} - \frac{25}{144} = \frac{36-25}{144} = \frac{11}{144}$

$\Rightarrow \sigma_x = \frac{\sqrt{11}}{12}$

Similarly $\sigma_y^2 = Var(y) = \frac{11}{144} \Rightarrow \sigma_y = \frac{\sqrt{11}}{12}$

$$E(XY) = \int_0^1\int_0^1 xy f(x,y)\,dx\,dy = \int_0^1\int_0^1 xy(2-x-y)\,dx\,dy$$

$$= \int_0^1\int_0^1 (2xy - x^2y - xy^2)\,dx\,dy = \int_0^1 \left(x^2y - \frac{x^3y}{3} - \frac{x^2y^2}{2} \right)_0^1 dy$$

$$= \int_0^1 \left(y - \frac{y}{3} - \frac{y^2}{2} \right) dy = \left[\frac{y^2}{2} - \frac{y^2}{6} - \frac{y^3}{6} \right]_0^1 = \frac{1}{2} - \frac{1}{6} - \frac{1}{6} = \frac{1}{2} - \frac{1}{3}$$

$$= \frac{3-2}{6} = \frac{1}{6}$$

$$Cov(X,Y) = E(XY) - E(X).E(Y) = \frac{1}{6} - \frac{5}{12}\cdot\frac{5}{12} = \frac{1}{6} - \frac{25}{144} = \frac{24-25}{144} = \frac{-1}{144}$$

The correlation coefficient is

$$r(X,Y) = \frac{Cov(X,Y)}{\sigma_X . \sigma_Y} = \frac{-1/144}{\frac{\sqrt{11}}{12}\cdot\frac{\sqrt{11}}{12}} = \frac{-1}{11}$$

③ The joint probability mass fun/. of X & Y is given below. Find correlation coefficient of (X,Y).

y \ x	-1	1
0	1/8	3/8
1	2/8	2/8

Sol:

y \ x	-1	1	$p(y) = p_{\cdot j}$
0	1/8	3/8	$P(Y=0) = 1/2$
1	2/8.	2/8	$P(Y=1) = 1/2$
$p(x) = p_{i\cdot}$	$P(X=-1)$ $= 3/8$	$P(X=1)$ $= 5/8$	

$$E(X) = \sum x_i P(x_i) = (-1)\left(\frac{3}{8}\right) + (1)\left(\frac{5}{8}\right) = \frac{-3}{8} + \frac{5}{8} = \frac{2}{8} = \frac{1}{4}$$

$$E(X^2) = \sum x_i^2 P(x_i) = (-1)^2\left(\frac{3}{8}\right) + (1)^2\left(\frac{5}{8}\right) = \frac{3}{8} + \frac{5}{8} = 1$$

$$E(Y) = \sum y_i P(y_i) = (0)(1/2) + 1(1/2) = 1/2$$

$$E(Y^2) = \sum y_i^2 P(y_i) = (0)^2(1/2) + (1)^2(1/2) = 1/2$$

$$E(XY) = \sum_i \sum_j x_i y_j P(x_i, y_j) = (0)(-1)\left(\frac{1}{8}\right) + (0)(1)\left(\frac{3}{8}\right) + (1)(-1)\left(\frac{2}{8}\right) + (1)(1)\left(\frac{2}{8}\right)$$

$$= \frac{-1}{4} + \frac{1}{4} = 0$$

$$\sigma_X^2 = Var(X) = E(X^2) - [E(X)]^2 = 1 - \left(\frac{1}{4}\right)^2 = 1 - \frac{1}{16} = \frac{15}{16}$$

$$\therefore \sigma_X = \frac{\sqrt{15}}{4}$$

$$\sigma_Y^2 = Var(Y) = E(Y^2) - [E(Y)]^2 = \frac{1}{2} - \left(\frac{1}{2}\right)^2 = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$$

$$\therefore \sigma_Y = \frac{1}{2}$$

$$r_{XY} = \frac{Cov(X,Y)}{\sigma_X \sigma_Y} = \frac{E(XY) - E(X)E(Y)}{\sigma_X \sigma_Y} = \frac{0 - \left(\frac{1}{4}\right)\left(\frac{1}{2}\right)}{\frac{\sqrt{15}}{4} \times \frac{1}{2}} = \frac{\frac{-1}{8}}{\frac{\sqrt{15}}{8}} = \frac{-1}{\sqrt{15}} = -0.2582$$

---

④ Suppose that the 2 dimensional rvs (X,Y) has the joint p.d.f.

$$f(x,y)=\begin{cases} x+y, & 0<x<1,\ 0<y<1 \\ 0, & \text{otherwise} \end{cases}$$

. Obtain the correlation coefficient between X & Y.

Sol: Marginal density fun. of X is given by

$$f(x)=\int_{-\infty}^{\infty} f(x,y)\,dy=\int_0^1 (x+y)\,dy=\left(xy+\frac{y^2}{2}\right)_0^1=x+\frac{1}{2},\quad 0<x<1$$

Marginal density fun. of Y is given by

$$f(y)=\int_{-\infty}^{\infty} f(x,y)\,dx=\int_0^1 (x+y)\,dx=\left(\frac{x^2}{2}+xy\right)_0^1=\frac{1}{2}+y,\quad 0<y<1$$

$$E(X)=\int_{-\infty}^{\infty} x f(x)\,dx=\int_0^1 x\left(x+\frac{1}{2}\right)dx=\int_0^1\left(x^2+\frac{x}{2}\right)dx=\left(\frac{x^3}{3}+\frac{x^2}{4}\right)_0^1$$

$$=\frac{1}{3}+\frac{1}{4}=\frac{4+3}{12}=\frac{7}{12}$$

Similarly, $E(Y)=\frac{7}{12}$

$$E(X^2)=\int_0^1 x^2 f(x)\,dx=\int_0^1 x^2(x+1/2)\,dx=\int_0^1\left(x^3+\frac{x^2}{2}\right)dx=\left(\frac{x^4}{4}+\frac{x^3}{6}\right)_0^1$$

$$=\frac{1}{4}+\frac{1}{6}=\frac{10}{24}=\frac{5}{12}$$

Similarly, $E(Y^2)=\frac{5}{12}$

$$E(XY)=\int_0^1\int_0^1 xy(x+y)\,dx\,dy=\int_0^1\int_0^1 (x^2y+xy^2)\,dx\,dy=\int_0^1\left(\frac{x^3y}{3}+\frac{x^2y^2}{2}\right)_0^1 dy$$

$$=\int_0^1\left(\frac{y}{3}+\frac{y^2}{2}\right)dy=\left(\frac{y^2}{6}+\frac{y^3}{6}\right)_0^1=\frac{1}{6}+\frac{1}{6}=\frac{1}{3}$$

$$Var(X)=E(X^2)-[E(X)]^2=\frac{5}{12}-\left(\frac{7}{12}\right)^2=\frac{5}{12}-\frac{49}{144}=\frac{60-49}{144}=\frac{11}{144}$$

$\therefore \sigma_X=\frac{\sqrt{11}}{12}$. Similarly $\sigma_Y=\frac{\sqrt{11}}{12}$

$$Cov(X,Y)=E(XY)-E(X)E(Y)=\frac{1}{3}-\left(\frac{7}{12}\right)\left(\frac{7}{12}\right)=\frac{1}{3}-\frac{49}{144}=\frac{48-49}{144}=\frac{-1}{144}$$

$$r_{XY}=\frac{Cov(X,Y)}{\sigma_X\cdot\sigma_Y}=\frac{-1/144}{\frac{\sqrt{11}}{12}\cdot\frac{\sqrt{11}}{12}}=\frac{-1/144}{11/144}=\frac{-1}{11}=-0.0909$$

---

⑤ Two independent random variables X & Y are defined by, $f(x)=\begin{cases}4ax, & 0\le x\le 1\\ 0, & \text{otherwise}\end{cases}$

$f(y)=\begin{cases}4by, & 0\le y\le 1\\ 0, & \text{otherwise}\end{cases}$. Show that $U=X+Y$ & $V=X-Y$ are uncorrelated.

Sol: Given $f(x)=\begin{cases}4ax, & 0\le x\le 1\\ 0, & \text{otherwise}\end{cases}$

$\because f(x)$ is the density fun. of X, $\int_0^1 f(x)\,dx=1 \Rightarrow \int_0^1 4ax\,dx=1$

$$\Rightarrow 4a\left(\frac{x^2}{2}\right)_0^1=1 \Rightarrow 4a\left(\frac{1}{2}\right)=1 \Rightarrow 2a=1 \Rightarrow a=1/2$$

$$f(y)=\begin{cases}4by, & 0\le y\le 1\\ 0, & \text{otherwise}\end{cases}$$

$\because f(y)$ is the density funct. of Y, $\int_0^1 f(y)dy=1 \Rightarrow \int_0^1 4by\,dy=1$

$$\Rightarrow 4b\left(\frac{y^2}{2}\right)_0^1=1 \Rightarrow 4b\left(\frac{1}{2}\right)=1 \Rightarrow 2b=1 \Rightarrow b=\frac{1}{2}$$

To prove $U=X+Y$ & $V=X-Y$ are uncorrelated. (ie) to prove $Cor(U,V)=0$.

$Cor(U,V)=E(UV)-E(U)E(V)$

$E(U)=E(X+Y)=E(X)+E(Y)$

$E(V)=E(X-Y)=E(X)-E(Y)$

$E(UV)=E[(X+Y)(X-Y)]=E(X^2-Y^2)=E(X^2)-E(Y^2)$

$$E(X)=\int_{-\infty}^{\infty}xf(x)dx=\int_0^1 x(2x)dx=2\int_0^1 x^2dx=2\left(\frac{x^3}{3}\right)_0^1=\frac{2}{3}$$

$$E(Y)=\int_{-\infty}^{\infty}yf(y)dy=\int_0^1 y(2y)dy=2\int_0^1 y^2dy=2\left(\frac{y^3}{3}\right)_0^1=\frac{2}{3}$$

$\times[E(XY)=E(X)E(Y)$ ($\because X$ & Y are independent)

$=\frac{2}{3}\cdot\frac{2}{3}=\frac{4}{9}]\times$

$E(U)=E(X)+E(Y)=\frac{2}{3}+\frac{2}{3}=\frac{4}{3}$

$E(V)=E(X)-E(Y)=\frac{2}{3}-\frac{2}{3}=0$

$E(UV)=E(X^2)-E(Y^2)=\frac{1}{2}-\frac{1}{2}=0$

$$E(X^2)=\int_{-\infty}^{\infty}x^2f(x)dx=\int_0^1 x^2(2x)dx$$

$$=\int_0^1 2x^3dx=2\left(\frac{x^4}{4}\right)_0^1=\frac{1}{2}$$

Similarly, $E(Y^2)=\frac{1}{2}$

$\therefore Cor(U,V)=E(UV)-E(U)E(V)$

$=0-\frac{4}{3}(0)=0$

Hence U & V are uncorrelated.

---

⑥ Find the rank correlation coefficient from the following data:

Rank in X :	1	2	3	4	5	6	7
Rank in Y :	4	3	1	2	6	5	7

Sol:

X	Y	$d_i=x_i-y_i$	d_i^2
1	4	−3	9
2	3	−1	1
3	1	2	4
4	2	2	4
5	6	−1	1
6	5	1	1
7	7	0	0

Here $n=7$, $\Sigma d_i^2=20$

$\therefore$ Rank correlation coefficient

$$r(x,y)=1-\frac{6\Sigma d_i^2}{n(n^2-1)}=1-\frac{(6\times20)}{7(49-1)}$$

$$=0.6429$$

⑦ Ten participants were ranked according to their performance in a musical test by the 3 judges in the following data:

	1	2	3	4	5	6	7	8	9	10
Rank by X :	1	6	5	10	3	2	4	9	7	8
Rank by Y :	3	5	8	4	7	10	2	1	6	9
Rank by Z :	6	4	9	8	1	2	3	10	5	7

Using rank correlation method, discuss which pair of judges has the nearest approach to common likings of music.

Sol:

x_i	y_i	z_i	$d_1 = x_i - y_i$	$d_2 = y_i - z_i$	$d_3 = x_i - z_i$	d_1^2	d_2^2	d_3^2
1	3	6	-2	-3	-5	4	9	25
6	5	4	1	1	2	1	1	4
5	8	9	-3	-1	-4	9	1	16
10	4	8	6	-4	2	36	16	4
3	7	1	-4	6	2	16	36	4
2	10	2	-8	8	0	64	64	0
4	2	3	2	-1	1	4	1	1
9	1	10	8	-9	-1	64	81	1
7	6	5	1	1	2	1	1	4
8	9	7	-1	2	1	1	4	1

The rank correlation coefficient between X & Y is given by

$$r(X,Y) = 1 - \frac{6\sum d_1^2}{n(n^2-1)} = 1 - \frac{(6\times 200)}{10(100-1)} = -0.2121$$

The rank correlation coefficient between Y & Z is given by

$$r(Y,Z) = 1 - \frac{6\sum d_2^2}{n(n^2-1)} = 1 - \frac{(6\times 214)}{(10\times 99)} = -0.297$$

The rank correlation coefficient between X & Z is given by

$$r(X,Z) = 1 - \frac{6\sum d_3^2}{n(n^2-1)} = 1 - \frac{(6\times 60)}{990} = 0.6364$$

Since the rank correlation coefficient between X & Z is positive & maximum, we conclude that the pair of judges X & Z has the nearest approach to common likings in music.

⑧ Obtain the rank correlation coefficient for the following data:

X :	68	64	75	50	64	80	75	40	55	64
Y :	62	58	68	45	81	60	68	48	50	70

Sol:

X	Y	Rank X (x_i)	Rank Y (y_i)	$d_i = x_i - y_i$	d_i^2
68	62	4	5	-1	1
64	58	6	7	-1	1
75	68	2.5	3.5	-1	1
50	45	9	10	-1	1
64	81	6	1	5	25
80	60	1	6	-5	25
75	68	2.5	3.5	-1	1
40	48	10	9	1	1
55	50	8	8	0	0
64	70	6	2	4	16

Correction factors:

In X series 75 repeated twice

$\therefore C.F. = \frac{2(2^2-1)}{12} = \frac{1}{2}$

In X series 64 repeated thrice

$\therefore C.F. = \frac{3(3^2-1)}{12} = 2$

In Y series 68 repeated twice

$\therefore C.F. = \frac{2(2^2-1)}{12} = \frac{1}{2}$

$\therefore$ Rank correlation $r = 1 - \frac{6\left(\Sigma d^2 + \frac{1}{2} + 2 + \frac{1}{2}\right)}{n(n^2-1)}$

$\Rightarrow r = 1 - \frac{6(72+3)}{10(10^2-1)}$

$= 0.5455$

Regression:

Regression is a mathematical measure of the average relationship between two or more variables interms of the original limits of the data.

Lines of regression:

(i) The line of regression of y on x is given by $y - \bar{y} = r\frac{\sigma_y}{\sigma_x}(x - \bar{x})$. — ①

(ii) The line of regression of x on y is given by $x - \bar{x} = r\frac{\sigma_x}{\sigma_y}(y - \bar{y})$. — ②

Regression coefficients:

(i) Regression coefficient of y on x is $r\frac{\sigma_y}{\sigma_x} = b_{yx}$

(ii) Regression coefficient of x on y is $r\frac{\sigma_x}{\sigma_y} = b_{xy}$

Correlation coefficient $r = \pm\sqrt{b_{yx}\, b_{xy}}$

where $b_{yx} = \frac{\Sigma(x-\bar{x})(y-\bar{y})}{\Sigma(x-\bar{x})^2}$; $b_{xy} = \frac{\Sigma(x-\bar{x})(y-\bar{y})}{\Sigma(y-\bar{y})^2}$

Properties of regression lines:

(i) The regression lines pass through $(\bar{x},\bar{y})$. So $(\bar{x},\bar{y})$ is the point of intersection of the regression lines.

(ii) When $r=1$, that is when there is a perfect +ve correlation or when $r=-1$, that is when there is a perfect -ve correlation the eqn. ① & ② becomes one are the same & so the regression lines coincide.

(iii) When $r=0$ the eqns. of the lines are $y=\bar{y}$ & $x=\bar{x}$ which represent perpendicular lines which are parallel to the axis.

(iv) The slopes of the lines are $r\frac{\sigma_y}{\sigma_x}$, $\frac{1}{r}\frac{\sigma_y}{\sigma_x}$. Since the S.D's σ_x & σ_y are +ve, both the slopes are +ve if r is +ve & -ve if r is -ve. That is all the three, namely the two slopes & r are of same sign.

Angle between the regression lines:

The slopes of the regression lines are $m_1 = r\frac{\sigma_y}{\sigma_x}$, $m_2 = \frac{1}{r}\frac{\sigma_y}{\sigma_x}$

If θ is the angle between the lines, then

$$\tan\theta = \frac{m_2-m_1}{1+m_1m_2} = \frac{\sigma_y}{\sigma_x}\frac{\frac{1}{r}-r}{1+\left(\frac{\sigma_y}{\sigma_x}\right)^2} = \frac{\sigma_x\sigma_y}{\sigma_x^2+\sigma_y^2}\left(\frac{1}{r}-r\right)$$

$$\Rightarrow \tan\theta = \frac{\sigma_x\sigma_y}{\sigma_x^2+\sigma_y^2}\left(\frac{1-r^2}{r}\right)$$

Note: (i) When $r=0$, that is, when there is no correlation between x & y.
$\tan\theta=\infty$ (or) $\theta=\frac{\pi}{2}$ & so the regression lines are perpendicular.

(ii) When $r=1$ or -1, that is, when their is a perfect correlation, +ve or -ve,
$\theta=0$ & so the lines coincide.

Correlation coefficient is the geometric mean between the two regression coefficients:

Proof: WKT $b_{xy} = r\frac{\sigma_x}{\sigma_y}$ & $b_{yx} = r\frac{\sigma_y}{\sigma_x}$

$$\Rightarrow (b_{xy})(b_{yx}) = r^2\frac{\sigma_x}{\sigma_y}\cdot\frac{\sigma_y}{\sigma_x} = r^2$$

$$\Rightarrow r = \pm\sqrt{(b_{xy})(b_{yx})}$$

If one of the regression coefficient is greater than unity the other must be less than unity:

Proof: WKT $r^2 = b_{xy}b_{yx} \le 1$ —① $\quad -1\le r\le 1 \Rightarrow r^2\le 1$

Assume that $b_{xy} > 1$

We have to prove that $b_{yx} < 1$

Since $b_{xy} > 1$; $\frac{1}{b_{xy}} < 1$

∴ ① ⇒ $b_{xy} b_{yx} \le 1$; $b_{yx} \le \frac{1}{b_{xy}} < 1$ ∴ $b_{yx} < 1$

Distinguish between correlation & regression Analysis:

Correlation	Regression
① Correlation means relationship between two variables.	- Regression is a mathematical measure of expressing the average relationship between the two variables.
② Correlation need not imply cause & effect relationship between the variables.	- Regression analysis clearly indicates the cause & effect relationship between variables.
③ Correlation coefficient is symmetric (ie) $r_{xy} = r_{yx}$.	- Regression coefficient is not symmetric. (ie) $b_{xy} \neq b_{yx}$
④ Correlation coefficient is a measure of the direction & degree of linear (length) relationship between two variables.	- Using the relationship between two variables we can predict the dependent variable value for any given independent variable value.

Standard errors of estimate:

The standard error of estimate of x is $S_x = \sigma_x \sqrt{1-r^2}$

The standard error of estimate of y is $S_y = \sigma_y \sqrt{1-r^2}$.

Correlation of grouped data:

When the no/. of observations is large & the variables are grouped, the data can be classified into two way frequency distribution called a correlation table. If there are 'n' classes for X & 'm' classes for Y, there will be (mxn) cells in the two-way table.

The formula for calculating the coefficient of correlation is $r = \frac{P}{\sigma_X \sigma_Y}$

where $P = \frac{\Sigma XY f_{XY}}{N} - \left(\frac{\Sigma X f_X}{N}\right)\left(\frac{\Sigma Y f_Y}{N}\right)$

$\sigma_X^2 = \frac{\Sigma X^2 f_X}{N} - \left(\frac{\Sigma X f_X}{N}\right)^2$ & $\sigma_Y^2 = \frac{\Sigma Y^2 f_Y}{N} - \left(\frac{\Sigma Y f_Y}{N}\right)^2$

Probable error of correlation coefficient:

The probable error of correlation coefficient is given by

$$P.E.(r) = 0.6745 \times S.E.$$

where S.E. is the standard error & is $S.E.(r) = \frac{1-r^2}{\sqrt{n}}$, where r is the correlation coefficient & n is the no/. of observation. Thus

$$P.E.(r) = 0.6745\left(\frac{1-r^2}{\sqrt{n}}\right)$$

The reason for taking the factor 0.6745 is that in a normal distribution, the range $\mu = \pm 0.6745$ covers 50% of the total area. This error enables us to find the limits within which correlation coefficient can be expected to vary.

Problems:

① From the following data, find (i) the two regression eqns/., (ii) the coefficient of correlation between the marks in Economics & Statistics, (iii) the most likely marks in Statistics when marks in Economics are 30.

Marks in Eco. x :	25	28	35	32	31	36	29	38	34	32
Statistics y :	43	46	49	41	36	32	31	30	33	39

Sol:

x	y	$x-\bar{x}$	$y-\bar{y}$	$(x-\bar{x})^2$	$(y-\bar{y})^2$	$(x-\bar{x})(y-\bar{y})$
25	43	-7	5	49	25	-35
28	46	-4	8	16	64	-32
35	49	3	11	9	121	33
32	41	0	3	0	9	0
31	36	-1	-2	1	4	2
36	32	4	-6	16	36	-24
29	31	-3	-7	9	49	21
38	30	6	-8	36	64	-48
34	33	2	-5	4	25	-10
32	39	0	1	0	1	0
320	380	0	0	140	398	-93

Here $\bar{x} = \frac{\Sigma x}{n} = \frac{320}{10} = 32$; $\bar{y} = \frac{\Sigma y}{n} = \frac{380}{10} = 38$

$$b_{yx} = \frac{\Sigma(x-\bar{x})(y-\bar{y})}{\Sigma(x-\bar{x})^2} = \frac{-93}{140} = -0.6643$$

$$b_{xy} = \frac{\Sigma(x-\bar{x})(y-\bar{y})}{\Sigma(y-\bar{y})^2} = \frac{-93}{398} = -0.2337$$

Eqn/. of the line of regression of x on y is

$$x-\bar{x} = b_{xy}(y-\bar{y}) \Rightarrow x-32 = -0.2337(y-38) \Rightarrow x = -0.2337y + 8.8806 + 32$$

$\Rightarrow x = -0.2337y + 40.8806$

Eqn. of the line of regression of y on x is

$$y - \bar{y} = b_{yx}(x - \bar{x}) \Rightarrow y - 38 = -0.6643(x - 32)$$

$$\Rightarrow y = -0.6643x + \overset{21.2576}{\cancel{32.6643}} + 38$$

$$\Rightarrow y = -0.6643x + \cancel{70.6643}\ 59.2576$$

Coefficient of correlation

$$r^2 = b_{yx} b_{xy} = (-0.6643)(-0.2337) = 0.1552$$

Since both regression coeff. are $-ve$, r must be $-ve$

$\Rightarrow r = \pm 0.394 \Rightarrow r = -0.394$

Now we have to find the most likely marks in statistics (y) when marks in Economics (x) are 30.

(ie) $y = -0.6643x + 59.2576$

When $x = 30 \Rightarrow y = 39$

---

② If the eqns. of the two lines of regression of y on x & x on y are respectively, $7x - 16y + 9 = 0$; $5y - 4x - 3 = 0$, calculate the coefficient of correlation, $\bar{x}$ & $\bar{y}$.

Sol: Since both the regression lines pass through $(\bar{x}, \bar{y})$, we get

$$7\bar{x} - 16\bar{y} + 9 = 0 \text{ --- ① } \qquad 5\bar{y} - 4\bar{x} - 3 = 0 \text{ --- ②}$$

$$① \times 4 \Rightarrow \cancel{28\bar{x}} - 64\bar{y} + 36 = 0$$

$$② \times 7 \Rightarrow -\cancel{28\bar{x}} + 35\bar{y} - 21 = 0$$

$$-29\bar{y} + 15 = 0 \Rightarrow \bar{y} = \frac{15}{29}$$

Subst. $\bar{y}$ value in ②, we get

$$5\left(\frac{15}{29}\right) - 4\bar{x} - 3 = 0 \Rightarrow 4\bar{x} = \frac{-12}{29} \Rightarrow \bar{x} = \frac{-3}{29}$$

∴ The mean values of x & y are $-\frac{3}{29}$ & $\frac{15}{29}$.

The regression eqn. of y on x is,

$$7x - 16y + 9 = 0 \Rightarrow 16y = +7x + 9 \Rightarrow y = \frac{7}{16}x + \frac{9}{16}$$

$$\therefore b_{yx} = \frac{7}{16}$$

Similarly, the regression eqn. of x on y is,

$$5y - 4x - 3 = 0 \Rightarrow 4x = 5y - 3 \Rightarrow x = \frac{5}{4}y - \frac{3}{4}$$

$$\therefore b_{xy} = \frac{5}{4}$$

Hence the correlation coefficient between X & Y is given by

$$r = \pm\sqrt{b_{xy} \times b_{yx}} = \pm\sqrt{\frac{5}{4} \times \frac{7}{16}} = \pm\sqrt{\frac{35}{64}} = \pm 0.7395$$

Since both the regressive coefficients are $+ve$, r must be $+ve$. $\therefore r = 0.7395$

⑧ Calculate the coefficient of correlation between x & y from the following table & write down the regression eqn. of y on x:

y \ x	0-	40-	80-	120-
10-	9	4	1	
30-	47	19	6	
50-	26	18	11	
70-	2	3	2	2

Sol:

Mid y \ Mid x	20	60	100	140	y-freq. f_y	$y=\frac{y-40}{20}$	yf_y	y^2f_y	xyf_{xy}
20	9	4	1		14	−1	−14	14	8
xyf_{xy}	9	0	−1						
40	47	19	6		72	0	0	0	0
xyf_{xy}	0	0	0						
60	26	18	11		55	1	55	55	−15
xyf_{xy}	−26	0	11						
80	2	3	2	2	9	2	18	36	8
xyf_{xy}	−4	0	4	8					
x-freq. f_x	84	44	20	2	150		59	105	1
$x=\frac{x-60}{40}$	−1	0	1	2					
xf_x	−84	0	20	4	−60				
x^2f_x	84	0	20	8	112				
xyf_{xy}	−21	0	14	8	1				

$\bar{x}=60,\ \bar{y}=40$

$$\sigma_x^2=\frac{\sum x^2f_x}{N}-\left(\frac{\sum xf_x}{N}\right)^2=\frac{112}{150}-\left(\frac{-60}{150}\right)^2=\frac{112}{150}-\frac{3600}{22500}=\frac{16800-3600}{22500}$$

$$=\frac{13200}{22500}=0.5867$$

$\therefore \sigma_x=0.766$

$$\sigma_y^2=\frac{\sum y^2f_y}{N}-\left(\frac{\sum yf_y}{N}\right)^2=\frac{105}{150}-\left(\frac{59}{150}\right)^2=\frac{105}{150}-\frac{3481}{22500}=\frac{15750-3481}{22500}$$

$$=\frac{12269}{22500}=0.5453$$

$\therefore \sigma_y=0.7384$

$$P=\frac{\sum xyf_{xy}}{N}-\left(\frac{\sum xf_x}{N}\right)\left(\frac{\sum yf_y}{N}\right)=\frac{1}{150}-\left(\frac{-60}{150}\right)\left(\frac{59}{150}\right)=\frac{1}{150}+\frac{3540}{22500}$$

$$=\frac{150+3540}{22500}=\frac{3690}{22500}=0.164$$

$$r = \frac{P}{\sigma_x \sigma_y} = \frac{0.164}{(0.766)(0.7384)} = 0.29$$

The regression eqn/. of y on x is

$$y - \bar{y} = r\frac{\sigma_y}{\sigma_x}(x - \bar{x}) \Rightarrow y - 40 = (0.29)\frac{0.7384}{0.766}(x - 60)$$

$$\Rightarrow y - 40 = 0.2796(x - 60) \Rightarrow y = 0.2796x - 16.776 + 40$$

$$\Rightarrow y = 0.2796x + 23.224$$

The regression eqn/. x on y is

$$x - \bar{x} = r\frac{\sigma_x}{\sigma_y}(y - \bar{y}) \Rightarrow x - 60 = (0.29)\frac{0.766}{0.7384}(y - 40)$$

$$\Rightarrow x - 60 = 0.3008(y - 40) \Rightarrow x = 0.3008y - 12.032 + 60$$

$$\Rightarrow x = 0.3008y + 47.968$$

④ For the following data find the most likely price at Madras corresponding to the price 70 at Bombay & that at Bombay corresponding to the price 68 at Madras.

	Madras	Bombay
Average price	65	67
S.D. of price	0.5	3.5

S.D. of the difference between the price at Madras & Bombay is 3.1

<u>Sol:</u> Let x denotes the price at Madras & y denotes the price at Bombay.

Given $\bar{x} = 65$; $\bar{y} = 67$, $\sigma_x = 0.5$, $\sigma_y = 3.5$, $\sigma_{x-y} = 3.1$

The correlation coefficient r is given by

$$r = \frac{\sigma_x^2 + \sigma_y^2 - \sigma_{x-y}^2}{2\sigma_x\sigma_y} = \frac{(0.5)^2 + (3.5)^2 - (3.1)^2}{2(0.5)(3.5)} = \frac{2.89}{3.5} = 0.8257$$

The line of regression of y on x is

$$y - \bar{y} = r\frac{\sigma_y}{\sigma_x}(x - \bar{x}) \Rightarrow y - 67 = (0.8257)\frac{3.5}{0.5}(x - 65)$$

$$\Rightarrow y - 67 = 5.7799(x - 65) \Rightarrow y = 5.7799x - 375.6935 + 68$$

$$\Rightarrow y = 5.7799x - 307.6935$$

Put $x = 68$

Then $y = 5.7799(68) - 307.6935 = 85.3397$

$\therefore$ Corresponding to the price 68 at Madras, the most likely price at Bombay is 85.34.

Similarly, the line of regression of x on y is

$$x - \bar{x} = r\frac{\sigma_x}{\sigma_y}(y - \bar{y}) \Rightarrow x - 65 = (0.8257)\frac{0.5}{3.5}(y - 67)$$

Put $y = 70$, then $x - 65 = (0.8257)\frac{0.5}{3.5}(70 - 67) \Rightarrow x = 65.3539$

$\therefore$ Corresponding to the price 70 at Bombay, the most likely price at Madras is 65.35.

Transformation of Random Variables:

Two funs/. of two random variables:

If (X,Y) is a 2-dimensional random variable with joint p.d.f. $f_{XY}(x,y)$ & if $Z=g(X,Y)$ & $W=h(X,Y)$ are two other rvs then the joint p.d.f. of (Z,W) is given by, $f_{ZW}(z,w)=\dfrac{f_{XY}(x,y)}{|J|}$ where $J=\begin{vmatrix}\frac{\partial z}{\partial x} & \frac{\partial z}{\partial y}\\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y}\end{vmatrix}$

Note: This result holds good, only if the eqn/. $Z=g(X,Y)$ & $W=h(X,Y)$ when solved, give unique values of x & y in terms of z & w.

One fun/. of two random variables:

If a rv Z is defined as $Z=g(X,Y)$, where X & Y are given rvs with joint p.d.f. $f(x,y)$. To find the p.d.f. of Z, we introduce a second random variable $W=h(X,Y)$ & obtain the joint p.d.f. of (Z,W), by using the previous result. Let it be $f_{ZW}(z,w)$. The required p.d.f. of Z is then obtained as the marginal p.d.f. is $f_Z(z)$ is obtained by simply integrating $f_{ZW}(z,w)$ w.r.t. w.

(ie) $f_Z(z)=\int_{-\infty}^{\infty} f_{ZW}(z,w)\,dw$

Problems:

① If X & Y are independent RVs with p.d.f. $e^{-x}, x\geq 0$; $e^{-y}, y\geq 0$ respectively. Find the density fun/. of $U=\frac{X}{X+Y}$ & $V=X+Y$. Are U & V independent?

Sol: Since X & Y are independent, $f_{XY}(x,y)=e^{-x}.e^{-y}=e^{-(x+y)}, x,y\geq 0$.

Solving the eqns/. $u=\frac{x}{x+y}$ & $v=x+y$, we get

$u=\frac{x}{v} \Rightarrow uv=x$; $y=v-x=v-uv=v(1-u)$

$x=uv$ & $y=v(1-u)$

$$J=\begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{vmatrix}=\begin{vmatrix}v & u\\ -v & 1-u\end{vmatrix}=v(1-u)+uv=v-uv+uv=v$$

The joint p.d.f. of (U,V) is given by,

$f_{UV}(u,v)=|J|\,f_{XY}(x,y)$

$=ve^{-(x+y)}=ve^{-(uv+v(1-u))}=ve^{-v}$ ①

The range space of (U,V) is obtained as follows:

$\because x,y\geq 0$, $uv\geq 0$ & $v(1-u)\geq 0$

∴ Either $u \geq 0, v \geq 0$ & $1-u \geq 0 \Rightarrow 1 \geq u$

(ie) $0 \leq u \leq 1$ & $v \geq 0$

(or) $u \leq 0, v \leq 0, 1-u \leq 0$ (ie) $u \leq 0, u \geq 1$ which is absurd.

∴ The range space of (u,v) is given by $0 \leq u \leq 1$ & $v \geq 0$.

∴ $f_{UV}(u,v) = ve^{-v}, 0 \leq u \leq 1$ & $v \geq 0$.

The p.d.f. of U is given by, $f_U(u) = \int_{-\infty}^{\infty} f_{UV}(u,v)dv = \int_{0}^{\infty} ve^{-v} dv$

$$\Rightarrow f_U(u) = \left[v.\frac{e^{-v}}{-1} - e^{-v}\right]_0^{\infty} = 1$$

(ie) U is uniformly distributed in (0,1).

The p.d.f. of V is given by $f_V(v) = \int_{-\infty}^{\infty} f_{UV}(u,v)du = \int_0^1 ve^{-v} du$

$$= ve^{-v}(u)_0^1 = ve^{-v}, v \geq 0.$$

Now, $f_U(u).f_V(v) = ve^{-v} = f_{UV}(u,v)$ (by ①)

⇒ U & V are independent rvs.

② If X & Y are independent rvs with density funs/. $f_X(x) = e^{-x}U(x)$ & $f_Y(y) = 2e^{-2y}U(y)$ $x,y \geq 0$. Find the density fun/. of $Z = X+Y$.

Sol: Since X & Y are independent, $f_{XY}(x,y) = f_X(x).f_Y(y)$

$$f_{XY}(x,y) = 2e^{-(x+2y)}, x,y \geq 0.$$

Let us consider the auxiliary rv, $W = Y$

∴ $x = z - w$ & $y = w$

$$J = \begin{vmatrix} \frac{\partial x}{\partial z} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial z} & \frac{\partial y}{\partial w} \end{vmatrix} = \begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} = 1$$

The joint p.d.f. of (Z,W) is given by,

$$f_{ZW}(z,w) = |J| f_{XY}(x,y) = 2e^{-(x+2y)} = 2e^{-(z-w+2w)} = 2e^{-(z+w)}.$$

The range space of (Z,W) is given as follows:

$w \geq 0, z - w \geq 0 \Rightarrow z \geq w$, $0 \leq w \leq z$

The p.d.f. of Z is given by,

$$f_Z(z) = \int_0^z f_{ZW}(z,w)dw = \int_0^z 2e^{-(z+w)}dw = 2e^{-z}\left[\frac{e^{-w}}{-1}\right]_0^z$$

$$= -2e^{-z}(e^{-z}-1) = 2e^{-z}(1-e^{-z}) = 2(e^{-z}-e^{-2z}), z \geq 0.$$

③ The joint p.d.f. of X & Y is given by $f(x,y) = e^{-(x+y)}, x>0, y>0$, find the probability density fun/. of $U = \frac{X+Y}{2}$.

Sol: Given $U = \frac{X+Y}{2}$ (ie) $u = \frac{x+y}{2}$

Let us make the transformation

$$u = \frac{1}{2}(x+y) \quad \& \quad v = y \quad \text{—①}$$

$\Rightarrow u \geq 0$ & $v \geq 0$ $(\because x>0, y>0)$

Also $u \geq v$ (ie) $u \geq 0$ & $0 \leq v \leq u$

From ①, we get, $x = 2u - v$, $y = v$

The jacobian J of the transformation is given by

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 2 & -1 \\ 0 & 1 \end{vmatrix} = 2$$

$2u - v > 0 \Rightarrow 2u > v$

$v > 0$

$0 < v < 2u \Rightarrow 2u > 0 \Rightarrow u > 0$

The joint p.d.f. of (u,v) is given by

$$f(u,v) = f(x,y)\,|J| = e^{-(x+y)} \cdot 2 = 2e^{-2u}$$

$u \geq 0$, $0 \leq v \leq u$ as $x+y = 2u$

The p.d.f. of u is given by

$$f_U(u) = \int_0^u f(u,v)\,dv = \int_0^u 2e^{-2u}\,dv = 2e^{-2u}(v)_0^u = 2ue^{-2u}, \quad u \geq 0.$$

---

④ If X & Y are independent rvs each normally distributed with mean zero & variance σ^2, find the density funs/. of $R = \sqrt{X^2+Y^2}$ & $\phi = \tan^{-1}\left(\frac{Y}{X}\right)$.

Sol: Since X & Y are independent rvs normally distributed with mean zero & variance σ^2, the joint p.d.f. of X & Y is given by

$$f_{XY}(x,y) = \frac{1}{2\pi\sigma^2} e^{\frac{-(x^2+y^2)}{2\sigma^2}}, \quad -\infty < x, y < \infty.$$

Given that $r = \sqrt{x^2+y^2}$ & $\theta = \tan^{-1}\left(\frac{y}{x}\right)$

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix}$$

Since the given transformation is a polar form of (x,y), we have

$$x = r\cos\theta \quad ; \quad y = r\sin\theta$$

$$\text{Hence } J = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r$$

The joint p.d.f. of (R,ϕ) is given by

$$f_{R\phi}(r,\theta) = f_{XY}(x,y)\,|J| = r \cdot \frac{1}{2\pi\sigma^2} e^{\frac{-(x^2+y^2)}{2\sigma^2}} = \frac{r}{2\pi\sigma^2} e^{\frac{-r^2}{2\sigma^2}}$$

Since $-\infty < x, y < \infty$, we have $0 \leq \theta \leq 2\pi$ & $0 < r < \infty$.

Hence $f_{R\phi}(r,\theta) = \frac{r}{\sigma^2 2\pi} e^{-r^2/2\sigma^2}$, $0 \leq \theta \leq 2\pi$, $0 < r < \infty$.

The p.d.f. of R is given by

$$f_R(r) = \int_0^{2\pi} f_{R\phi}(r,\theta)\,d\theta = \frac{r}{2\pi\sigma^2} e^{-r^2/2\sigma^2} \int_0^{2\pi} d\theta = \frac{r}{2\pi\sigma^2} e^{\frac{-r^2}{2\sigma^2}} \cdot 2\pi = \frac{r}{\sigma^2} e^{\frac{-r^2}{2\sigma^2}}, \quad 0 < r < \infty.$$

$$f_\phi(\theta) = \int_0^\infty f_{R\phi}(r,\theta)\,dr = \int_0^\infty \frac{r}{2\pi\sigma^2} e^{-r^2/2\sigma^2} dr = \frac{1}{2\pi\sigma^2} \int_0^\infty r e^{-r^2/2\sigma^2} dr$$

Take $u = \frac{r^2}{2\sigma^2} \Rightarrow du = \frac{1}{2\sigma^2} 2r\,dr = \frac{r}{\sigma^2} dr \Rightarrow r\,dr = \sigma^2 du$

$$\therefore f_\phi(\theta) = \frac{1}{2\pi\sigma^2} \int_0^\infty e^{-u} \sigma^2 du = \frac{1}{2\pi} \int_0^\infty e^{-u} du = \frac{1}{2\pi} \left[\frac{e^{-u}}{-1}\right]_0^\infty = \frac{-1}{2\pi}(0-1) = \frac{1}{2\pi}, \quad 0 \leq \theta \leq 2\pi$$

Central Limit Theorem:

Liapounoff's form:

If X_i $(i = 1, 2, \ldots, n)$ be independent random variables such that $E(X_i) = \mu_i$ & $Var(X_i) = \sigma_i^2$ then under certain general conditions, the rv $S_n = X_1 + X_2 + \ldots + X_n$ is asympototically normal with mean μ & standard deviation σ where $\mu = \sum_{i=1}^n \mu_i$ & $\sigma^2 = \sum_{i=1}^n \sigma_i^2$ as $n \to \infty$.

Proof: $M_{X_i}(t) = E(e^{tX_i})$

$= e^{t.1} p + e^{t.0} q = q + pe^t$ — ①

$M_{S_n}(t) = M_{X_1 + X_2 + \cdots + X_n}(t)$

$= M_{X_1}(t) M_{X_2}(t) \ldots M_{X_n}(t)$

$= (M_{X_i}(t))^n$ (As X_i's are independent & identically distributed)

$= (q + pe^t) \ldots n$ times

$= (q + pe^t)^n$ (by ①) — ② which is m.g.f. of a binomial variate with parameters n & p.

Hence by uniqueness thm. of m.g.f.

$S_n \approx B(n,p)$, $B(n,p)$ is the Binomial distribution.

$\therefore E(S_n) = np = \mu$ (say)

$Var(S_n) = npq = \sigma^2$ (say)

Let $Z = \frac{S_n - E(S_n)}{\sqrt{Var(S_n)}} = \frac{S_n - \mu}{\sigma}$

Then $M_Z(t) = e^{\frac{-\mu t}{\sigma}} M_{S_n}(t/\sigma)$

$$= e^{\frac{-npt}{\sqrt{npq}}}\left(q+pe^{\frac{t}{\sqrt{npq}}}\right)^n \quad (\text{By } ②)$$

$$= \left[1+\frac{t^2}{2n}+O(n^{-3/2})\right]^n$$

where $O(n^{-3/2})$ represents terms involving $n^{-3/2}$ & higher powers of n in the denominator.

As $n \to \infty$, we get, $\lim\limits_{n\to\infty} M_Z(t) = \lim\limits_{n\to\infty}\left[1+\frac{t^2}{2n}+O(n^{-3/2})\right]^n$

$= \lim\limits_{n\to\infty}\left(1+\frac{t^2}{2n}\right)^n = e^{t^2/2}$ which is the m.g.f. of the standard normal variable.

Hence $S_n = X_1 + X_2 + \cdots + X_n$ is asymptotically equivalent to $N(\mu, \sigma^2)$ as $n \to \infty$.

Lindberg - Levy's form:

If $X_1, X_2, \ldots, X_n, \ldots$ be a sequence of independent identically distributed rvs with $E(X_i) = \mu$ & $Var(X_i) = \sigma^2$, $i = 1, 2, \ldots$ & if $S_n = X_1 + X_2 + \cdots + X_n$, then under certain general conditions, S_n follows a normal distribution with mean $\underline{n\mu}$ & variance $\underline{n\sigma^2}$ as $n \to \infty$.

Corollary:

If $\bar{X} = \frac{1}{n}(X_1 + X_2 + \cdots + X_n)$, then $E(\bar{X}) = \mu$ & $Var(\bar{X}) = \frac{1}{n^2}(n\sigma^2) = \frac{\sigma^2}{n}$

$\therefore \bar{X}$ follows a normal distribution with mean μ & variance $\frac{\sigma^2}{n}$ as $n \to \infty$.

Applications of Central Limit Theorem:

(i) This thm. provides a simple method for computing approximate probabilities of sums of independent random variables.

(ii) It also gives us the wonderful fact that the empirical frequencies of so many natural populations exhibit a bell shaped curve. (ie, a normal curve).

Problems:

① The lifetime of a certain brand of an electric bulb may be considered as a rv with mean 1200h & standard deviation 250h. Find the probability, using central limit theorem, that the average lifetime of 60 bulbs exceeds 1250h.

Sol: If X_i denotes the lifetime of the light, then we have

Mean $= E(X_i) = 1200 = \mu$; Variance $= Var(X_i) = 250^2 = \sigma^2$

Let us assume that $\bar{x}$ denote the mean lifetime of 60 lights.

By Corollary of Lindeberg-Levy's form of Central limit thm., we have $\bar{x}$ follows a normal distribution with mean μ & variance $\frac{\sigma^2}{n}$.

N(mean, S.D)

(ie) $\bar{x}$ follows $N\left(\mu, \frac{\sigma}{\sqrt{n}}\right) \Rightarrow \bar{x}$ follows $N\left(1200, \frac{250}{\sqrt{60}}\right)$

We have to find the probability of the average lifetime of 60 lights exceeds 1250h.

(ie) to find $P(\bar{x} > 1250)$.

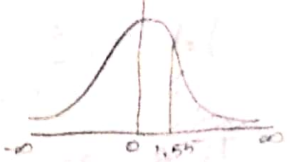
$$z = \frac{\bar{x} - \mu}{S.D}$$ variable, mean

Let $z = \frac{\bar{x}-\mu}{\sigma/\sqrt{n}}$, z a standard normal variable.

$$= \frac{\bar{x}-1200}{250/\sqrt{60}}$$

Now, $P(\bar{x} > 1250) = P\left(\frac{\bar{x}-1200}{250/\sqrt{60}} > \frac{1250-1200}{250/\sqrt{60}}\right) = P\left(z > \frac{50\times\sqrt{60}}{250}\right)$

$= P(z > 1.55)$ ~~$= P(0<z<\infty) - P(0<z<1.55)$~~



$P(0<z<1.55)$

$= 0.5 -$ (Area from 0 to 1.55)

$= 0.5 - 0.4394$ (from the area normal table)

$= 0.0606$

---

② If $x_1, x_2, \dots, x_n$ are Poisson variates with parameter $\lambda = 2$, use the central limit thm. to estimate $P(120 \le S_n \le 160)$, where $S_n = x_1 + x_2 + \dots + x_n$ & $n = 75$.

Sol: Given that $E(x_i) = \lambda = 2 = \mu$ & $Var(x_i) = \lambda = 2 = \sigma^2$

($\because$ For Poisson distribution Mean = Variance = λ], $i = 1, 2, \dots, n$.

By Central limit thm., we have S_n follows a normal distribution with mean $n\mu$ & variance $n\sigma^2$. (ie) S_n follows $N(n\mu, \sigma\sqrt{n})$.

Also $n = 75$.

Hence S_n follows $N(75\times 2, \sqrt{2}\times\sqrt{75})$

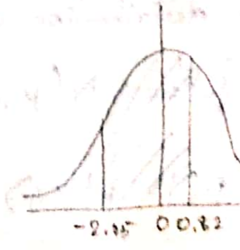
$\Rightarrow S_n$ follows $N(150, \sqrt{150})$

To find $P(120 \le S_n \le 160)$

Let $z = \frac{S_n - n\mu}{\sigma\sqrt{n}} = \frac{S_n - 150}{\sqrt{150}}$, z is a standard normal variable.

If $S_n = 120$, $z = \frac{120-150}{\sqrt{150}} = \frac{-30}{\sqrt{150}} = -2.45$ —①

If $S_n = 160$, $z = \frac{160-150}{\sqrt{150}} = \frac{10}{\sqrt{150}} = 0.82$



Now, $P(120 \le S_n \le 160) = P\left(\frac{120-150}{\sqrt{150}} \le z \le \frac{160-150}{\sqrt{150}}\right)$

$= P(-2.45 \le z \le 0.82)$

$= P(0 \leq Z \leq 2.45) + P(0 \leq Z \leq 0.82)$

$= 0.4929 + 0.2939 = 0.7868$

---

③ Let $X_1, X_2, \ldots, X_{100}$ be independent identically distributed random variables with $\mu = 2$ & $\sigma^2 = \frac{1}{4}$. Find $P(192 < X_1 + X_2 + \cdots + X_{100} < 210)$.

Sol: Given that $E(X_i) = \mu = 2$ & $Var(X_i) = \frac{1}{4} = \sigma^2$, $i = 1, 2, \ldots, 100$.

By Central limit thm., we have S_n follows a normal distribution with mean $n\mu$ & variance $n\sigma^2$, $S_n = X_1 + X_2 + \cdots + X_{100}$.

(ie) S_n follows $N(n\mu, \sigma\sqrt{n})$

Hence S_n follows $N(100 \times 2, \frac{1}{2}\sqrt{100}) \Rightarrow S_n$ follows $N(200, 5)$.

To find $P(192 < S_n < 210)$

Let $Z = \frac{S_n - n\mu}{\sigma\sqrt{n}} = \frac{S_n - 200}{5}$, Z is a standard normal variable.

If $S_n = 192$, $Z = \frac{192 - 200}{5} = \frac{-8}{5} = -1.6$ — ①

If $S_n = 210$, $Z = \frac{210 - 200}{5} = \frac{10}{5} = 2$ — ②

Now, $P(192 < S_n < 210) = P\left(\frac{S_n - 200}{5} < Z < \frac{S_n + 200}{5}\right)$

$= P(-1.6 < Z < 2)$ ($\because$ By ① & ②)

$= P(-1.6 < Z < 0) + P(0 < Z < 2)$

$= P(0 < Z < 1.6) + P(0 < Z < 2)$

$= 0.4452 + 0.4772 = 0.9224$

---

④ A random sample of size 100 is taken from a population whose mean is 60 & variance is 400. Using Central limit thm., with what probability can we assert that the mean of the sample will not differ from $\mu = 60$ by more than 4.

Sample mean $= \bar{X}$

Sol: Given that $n = 100$, $\mu = 60$, $\sigma^2 = 400$.

By the corollary of Lindeberg-Levy form of Central limit thm., $\bar{X}$ follows a normal distribution with mean μ & variance $\frac{\sigma^2}{n}$.

(ie) $\bar{X}$ follows $N\left(\mu, \frac{\sigma}{\sqrt{n}}\right) \Rightarrow \bar{X}$ follows $N\left(60, \frac{20}{\sqrt{100}}\right)$.

To find $P(|\bar{X} - \mu| \leq 4)$

Now, $P(|\bar{X} - \mu| \leq 4) = P(-4 \leq \bar{X} - \mu \leq 4) = P(-4 \leq \bar{X} - 60 \leq 4)$

$= P(56 \leq \bar{x} \leq 64)$

Let $z = \frac{\bar{x}-\mu}{\sigma/\sqrt{n}} = \frac{\bar{x}-60}{20/\sqrt{100}} = \frac{\bar{x}-60}{2}$

When $\bar{x} = 56$, $z = \frac{56-60}{2} = -2$ ①

When $\bar{x} = 64$, $z = \frac{64-60}{2} = 2$ ②

Hence, $P(56 \leq \bar{x} \leq 64) = P\left(\frac{56-60}{2} \leq \frac{\bar{x}-60}{2} \leq \frac{64-60}{2}\right)$

$= P(-2 \leq z \leq 2)$ (∵ by ① & ②)

$= P(-2 \leq z \leq 0) + P(0 \leq z \leq 2)$

$= P(0 \leq z \leq 2) + P(0 \leq z \leq 2) = 2P(0 \leq z \leq 2)$

$= 2(0.4772) = 0.9544$

---

⑤ A distribution with unknown mean μ has variance equal to 1.5. Use Central limit thm. to determine how large a sample should be taken from the distribution in order that the probability will be atleast 0.95 that the sample mean will be within 0.5 of the population mean.

Sol: Let n be the size of the sample.

Given $E(x) = \mu = $ mean, $Var(x) = \sigma^2 = 1.5$

Let $\bar{x}$ be the sample mean. By Corollary of Lindberg-Levy's form of Central limit thm., we have $\bar{x}$ follows a normal distribution with mean μ & variance $\frac{\sigma^2}{n}$.

(ii) $\bar{x}$ follows $N\left(\mu, \frac{\sigma}{\sqrt{n}}\right) \Rightarrow \bar{x}$ follows $N\left(\mu, \frac{\sqrt{1.5}}{\sqrt{n}}\right)$

To find n such that $P(|\bar{x}-\mu| < 0.5) \geq 0.95$.

Consider, $P(|\bar{x}-\mu| < 0.5) \geq 0.95$

$\Rightarrow P(-0.5 < \bar{x}-\mu < 0.5) \geq 0.95$

Let $z = \frac{\bar{x}-\mu}{\sigma/\sqrt{n}} \doteq \frac{\bar{x}-\mu}{\sqrt{1.5}/\sqrt{n}}$, z a standard normal variable.

$\therefore P(-0.5 < \bar{x}-\mu < 0.5) \geq 0.95$

$\Rightarrow P\left(\frac{-0.5\sqrt{n}}{\sqrt{1.5}} < z < \frac{0.5\sqrt{n}}{\sqrt{1.5}}\right) \geq 0.95$

$\Rightarrow P(|z| < 0.4082\sqrt{n}) \geq 0.95$

The least value of n is obtained from $P(|z| < 0.4082\sqrt{n}) = 0.95$

$P(-0.4082\sqrt{n} < z < 0.4082\sqrt{n}) = 0.95$

$\Rightarrow P(-0.4082\sqrt{n} < z < 0) + P(0 < z < 0.4082\sqrt{n}) = 0.95$

$\Rightarrow 2P(0 < z < 0.4082\sqrt{n}) = 0.95$

$\Rightarrow P(0 < z < 0.4082\sqrt{n}) = 0.475$

From the table of areas under normal curve,

$P(|z| < 1.96) = 0.95$

$\Rightarrow 0.4082\sqrt{n} = 1.96$

$\Rightarrow \sqrt{n} = \dfrac{1.96}{0.4082} \Rightarrow n = \left(\dfrac{1.96}{0.4082}\right)^2 \Rightarrow n = 23$

$\therefore$ The size of the sample must be atleast 23.

Central limit theorem: (Laplace discovered)

Let $x_1, x_2, \ldots$ be a sequence of independent & identically distributed rvs each having mean μ & variance σ^2. Then the distribution of $\dfrac{x_1 + x_2 + \cdots + x_n - n\mu}{\sigma\sqrt{n}}$ tends to the standard normal as $n \to \infty$. That is, for $-\infty < a < \infty$, $P\left\{\dfrac{x_1 + \cdots + x_n - n\mu}{\sigma\sqrt{n}} \leq a\right\} \to \dfrac{1}{\sqrt{2\pi}}\int_{-\infty}^{a} e^{-x^2/2}\,dx$ as $n \to \infty$.

Liapounoff's Form:

If $x_1, x_2, \ldots, x_n$ be a sequence of independent rvs with $E(x_i) = \mu_i$ & $Var(x_i) = \sigma_i^2$, $i = 1, 2, \ldots, n$ & if $S_n = x_1 + x_2 + \cdots + x_n$ then under certain general conditions, S_n follows a normal distribution with mean $\mu = \sum_{i=1}^{n} \mu_i$ & variance $\sigma^2 = \sum_{i=1}^{n} \sigma_i^2$ as $n \to \infty$.

Lindberg-Levy's Form:

If $x_1, x_2, \ldots, x_n$ be a sequence of independent identically distributed rvs with $E(x_i) = \mu$ & $Var(x_i) = \sigma^2$, $i = 1, 2, \ldots, n$ & if $S_n = x_1 + x_2 + \cdots + x_n$, then under certain general conditions, S_n follows a normal distribution with mean $n\mu$ & variance $n\sigma^2$ as $n \to \infty$.

Central Limit Theorem: [Lindberg-Levy's form]

Statement:

If $X_1, X_2, \ldots, X_n$ be a sequence of independent identically distributed random variables with $E(X_i) = \mu$ & $Var(X_i) = \sigma^2, i = 1, 2, \ldots, n$ & if $S_n = X_1 + X_2 + \cdots + X_n$, then under certain general conditions, S_n follows a normal distribution with mean $n\mu$ & variance $n\sigma^2$ as $n \to \infty$.

Proof: Given:

(a) $X_1, X_2, \ldots, X_n$ be n independent & identically distributed r.v.s.

(b) $E(X_1) = E(X_2) = \cdots = E(X_n) = \mu$

(c) $Var(X_1) = Var(X_2) = \cdots = Var(X_n) = \sigma^2$

(d) $S_n = X_1 + X_2 + \cdots + X_n$

To prove: (1) Mean of $S_n = n\mu$

(2) $Var(S_n) = n\sigma^2$

(3) S_n must be a normal variate with mean $n\mu$ & standard deviation $\sigma\sqrt{n}$.

(i) to prove : $M_Z(t) = e^{t^2/2}$ as $n \to \infty$, where $Z = \frac{S_n - n\mu}{\sigma\sqrt{n}}$

(1) Mean of $S_n = E(S_n) = E(X_1 + X_2 + \cdots + X_n)$

$$= E(X_1) + E(X_2) + \cdots + E(X_n) \quad (\because \text{by } (a))$$

$$= \mu + \mu + \cdots + \mu \quad (\because \text{by } (b))$$

$$= n\mu$$

(2) $Var(S_n) = Var(X_1 + X_2 + \cdots + X_n)$

$$= Var(X_1) + Var(X_2) + \cdots + Var(X_n) \quad (\because \text{by } (a))$$

$$= \sigma^2 + \sigma^2 + \cdots + \sigma^2 \quad (\because \text{by } (c))$$

$$= n\sigma^2$$

(3) Here, $Z = \frac{S_n - n\mu}{\sigma\sqrt{n}}$

$$M_Z(t) = E[e^{tZ}] = E\left[e^{t\left(\frac{S_n - n\mu}{\sigma\sqrt{n}}\right)}\right]$$

$$= \left[E\left[e^{t\left(\frac{x_1-\mu}{\sigma\sqrt{n}}\right)} \right] \right]^n \text{ — ① } (\because \text{by } (a))$$

$$E\left[e^{t\left(\frac{x_1-\mu}{\sigma\sqrt{n}}\right)} \right] = E\left[1 + \frac{\left(\frac{x_1-\mu}{\sigma\sqrt{n}}\right)}{1!} t + \frac{\left(\frac{x_1-\mu}{\sigma\sqrt{n}}\right)^2}{2!} t^2 + \cdots \right]$$

$$= E(1) + \frac{t}{\sigma\sqrt{n}} E(x_1-\mu) + \frac{t^2}{2\sigma^2 n} E(x_1-\mu)^2 + \cdots$$

$$= 1 + \frac{t}{\sigma\sqrt{n}} \mu_1 + \frac{t^2}{2\sigma^2 n} \mu_2 + \cdots \quad (\because \mu_r = E(x-\bar{x})^r)$$

$$= 1 + 0 + \frac{t^2}{2n} + \cdots \quad (\because \mu_1 = 0, \mu_2 = \sigma^2)$$

$$= 1 + \frac{t^2}{2n} + \cdots$$

$$\therefore ① \Rightarrow M_z(t) = \left(1 + \frac{t^2}{2n} + \cdots\right)^n$$

$$\lim_{n\to\infty} M_z(t) = \lim_{n\to\infty} \left(1 + \frac{t^2}{2n} + \cdots\right)^n$$

$$= e^{t^2/2}$$

$$\therefore \lim_{n\to\infty} M_z(t) = e^{t^2/2}$$

∴ By using uniqueness property of m.g.f., the variate, Z must be a standard normal variate as $n \to \infty$.

$$\therefore Z = \frac{S_n - n\mu}{\sigma\sqrt{n}}$$

∴ S_n must be a normal variate having (mean $= n\mu$) & (s.d $= \sigma\sqrt{n}$)

Thus, as $n \to \infty$, $S_n \sim N[n\mu, n\sigma^2]$

Hence the proof.

Result 1:

$$E\left[e^{t\left(\frac{S_n - n\mu}{\sigma\sqrt{n}}\right)} \right] = \left[E\left(e^{t\left(\frac{x_1-\mu}{\sigma\sqrt{n}}\right)} \right) \right]^n$$

Proof:

$$E\left[e^{t\left(\frac{S_n-n\mu}{\sigma\sqrt{n}}\right)}\right] = E\left[e^{\frac{t[(x_1+x_2+\cdots+x_n)-(\mu+\mu+\cdots+\mu)]}{\sigma\sqrt{n}}}\right]$$

$$= E\left[e^{t\left[\frac{x_1-\mu}{\sigma\sqrt{n}}+\frac{x_2-\mu}{\sigma\sqrt{n}}+\cdots+\frac{x_n-\mu}{\sigma\sqrt{n}}\right]}\right]$$

$$= E\left[e^{t\left(\frac{x_1-\mu}{\sigma\sqrt{n}}\right)}\cdot e^{t\left(\frac{x_2-\mu}{\sigma\sqrt{n}}\right)}\cdots e^{t\left(\frac{x_n-\mu}{\sigma\sqrt{n}}\right)}\right]$$

$$= E\left[e^{t\left(\frac{x_1-\mu}{\sigma\sqrt{n}}\right)}\right] E\left[e^{t\left(\frac{x_2-\mu}{\sigma\sqrt{n}}\right)}\right]\cdots E\left[e^{t\left(\frac{x_n-\mu}{\sigma\sqrt{n}}\right)}\right]$$

$(\because$ by (a))

$$= \left[E\left[e^{t\left(\frac{x_1-\mu}{\sigma\sqrt{n}}\right)}\right]\right]^n$$

Result 2:

$$\lim_{n\to\infty} M_Z(t) = \lim_{n\to\infty}\left[1+\frac{t^2}{2n}+\cdots\right]^n = e^{t^2/2}$$

Let $u = \lim_{n\to\infty}\left[1+\frac{t^2}{2n}+\cdots\right]^n$

$$\log u = \lim_{n\to\infty} n\log\left[1+\frac{t^2}{2n}+\cdots\right]$$

$$= \lim_{n\to\infty}\frac{\log\left(1+\frac{t^2}{2n}+\cdots\right)}{1/n} = \frac{\infty}{\infty} \text{ (Indeterminate form)}$$

$$= \lim_{n\to\infty}\frac{\left(\frac{1}{1+\frac{t^2}{2n}+\cdots}\right)\left(0-\frac{t^2}{2n^2}+\cdots\right)}{\frac{-1}{n^2}}$$

by L'Hospital rule.

$$= \lim_{n\to\infty}\left[\left(\frac{1}{1+\frac{t^2}{2n}+\cdots}\right)\left(\frac{t^2}{2}+\cdots \text{terms in higher powers of } \frac{1}{n}\right)\right]$$

$$= \left(\frac{1}{1+0+0+\cdots}\right)\left(\frac{t^2}{2}+0+0+\cdots\right)$$

$$= \frac{t^2}{2} \qquad \therefore u = e^{t^2/2}$$

Central limit theorem (Lindberg-Levy's form)

If $X_1, X_2, \cdots, X_n$ be a sequence of independent identically distributed random variables with $E(X_i) = \mu$ & $Var(X_i) = \sigma^2$, $i = 1, 2, \cdots, n$ & if $S_n = X_1 + X_2 + \cdots + X_n$, then under certain general conditions, S_n follows a normal distribution with mean $n\mu$ & variance $n\sigma^2$ as $n \to \infty$.

Uses of Central Limit theorem:

① It is very useful in statistical surveys for a large sample size. It helps to provide fairly accurate results.

② It states that almost all theoretical distributions converge to normal distribution as $n \to \infty$.

③ It helps to find out the distribution of the sum of a large number of independent random variables.

④ It also helps explain the remarkable fact that the emprical frequencies of so many natural populations exhibit bell shaped (ie, normal) curves.

Twenty dice are thrown. Find approximately the prob. that the sum obtained is between 65 & 75 using CLT.

Sol: $\mu = E(X) = \sum_{i=1}^{6} i\left(\frac{1}{6}\right) = \frac{1}{6}\sum_{i=1}^{6} i$

$= \frac{1}{6}\left[1+2+3+4+5+6\right] = 3.5$

$E(X^2) = \sum_{i=1}^{6} i^2\left(\frac{1}{6}\right) = \frac{1}{6}\sum_{i=1}^{6} i^2$

$= \frac{1}{6}\left[1+4+9+16+25+36\right] = \frac{91}{6}$

$\sigma^2 = Var(X) = E(X^2) - \left[E(X)\right]^2 = \frac{91}{6} - (3.5)^2 = 2.92$

$\sigma = \sqrt{2.92} = 1.71$

$n = 20$

$S_n \rightarrow$ Sample sum

$S_n = X_1 + X_2 + \dots + X_{20}$

If the sum of random variables follows normal distribution, then S_n follows $N(n\mu, \sigma\sqrt{n})$ by CLT

$$Z = \frac{S_n - n\mu}{\sigma\sqrt{n}}$$

$$n\mu = 20 \times 3.5 = 70$$

$$\sigma\sqrt{n} = 1.71 \times \sqrt{20} = 7.65$$

$$Z = \frac{S_n - 70}{7.65}$$

To find: $P[65 < S_n < 75]$

when $S_n = 65$, $Z = \frac{65-70}{7.65} = -0.65$

when $S_n = 75$, $Z = \frac{75-70}{7.65} = 0.65$

$$P[65 < S_n < 75] = P[-0.65 < Z < 0.65]$$

$$= 2P[0 < Z < 0.65]$$

$$= 2 \times 0.2422$$

$$= 0.4844$$

RANDOM PROCESSES

Random Process:

A random process is a collection of random variables $\{X(s,t)\}$ that are funs. of a real variable, namely time t where $s \in S$ (Sample space) & $t \in T$ (parameter set or index set).

State space:

The set of possible values of any individual member of the random process is called state space.

Any individual member itself is called a sample fun. or a realisation of the processes.

Classification of process:

(i) Continuous random process:

If both X & t are continuous the random process is called as continuous random process. E.g: If $X(t)$ represents the maximum temperature at a place in the interval $(0,t)$, $\{X(t)\}$ is a continuous random process.

(ii) Continuous random sequence:

If X is continuous & t is discrete, the random process is called as continuous random sequence. E.g: If X_n represents the temperature at the end of the n^{th} hour of a day, then $\{X_n, 1 \le n \le 24\}$ is a continuous random sequence, since temperature can take any value in the interval & hence ~~continuous~~. discrete.

(iii) Discrete random process:

If X is discrete & t is continuous, the random process is called as discrete random process. E.g: If $X(t)$ represents the no. of telephone calls received in the interval $(0,t)$ then $\{X(t)\}$ is a discrete random process since $S = \{0,1,2,3,\dots\}$.

(iv) Discrete random sequence:

If both X & t are discrete, then the random process is called as discrete random sequence. E.g: If X_n represents the outcome of the

n^{th} toss of a fair die, then $\{X_n : n \geq 1\}$ is a discrete random sequence. Since $T = \{1,2,3,\ldots\}$ & $S = \{1,2,3,4,5,6\}$.

Comparison of Random variable & Random process:

Random variable	Random process
① A fun/. of the possible outcomes of an experiment (ie) $X(S)$.	A fun/. of the possible outcomes of an experiment & also time. (ie) $X(s,t)$.
② Outcome is mapped into a no/. x.	Outcomes are mapped into wave form which is a fun/. of time t.

Stationary process (or) Strictly stationary process (or) Strict sense stationary process:

A random process $X(t)$ is said to be stationary in the strict sense if its statistical characteristics do not change with time.

(ie) the random processes $X(t_1)$ & $X(t_2)$ where $t_2 = t_1 + \Delta$ will have all statistical properties the same.

Jointly stationary in the strict sense:

Two real-valued random processes $\{X(t)\}$ & $\{Y(t)\}$ are said to be jointly stationary in the strict sense, if the joint distribution of $X(t)$ & $Y(t)$ are invariant under translation of time.

First order stationary process:

A random process is called stationary to order one. If its first-order density fun/. does not change with a shift in time origin.

A random process that is not stationary in any sense is called an evolutionary process.

Stationary process:

Formula: $E[X(t)] = \text{Constant}$ & $V[X(t)] = \text{Constant}$.

Problems:

① Consider the random process $X(t) = \cos(t+\phi)$, where ϕ is a random variable with density func. $f(\phi) = \frac{1}{\pi}$, $-\frac{\pi}{2} < \phi < \frac{\pi}{2}$, check whether the process is stationary or not.

Sol: Given $X(t) = \cos(t+\phi)$ & $f(\phi) = \frac{1}{\pi}$, $-\frac{\pi}{2} < \phi < \frac{\pi}{2}$.

$$E[X(t)] = \int_{-\pi/2}^{\pi/2} X(t) f(\phi) d\phi = \int_{-\pi/2}^{\pi/2} \cos(t+\phi) \cdot \frac{1}{\pi} d\phi$$

$$= \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} \cos(t+\phi) d\phi = \frac{1}{\pi} \Big[\sin(t+\phi)\Big]_{-\pi/2}^{\pi/2}$$

$$= \frac{1}{\pi}\left[\sin\left(\frac{\pi}{2}+t\right) - \sin\left(-\frac{\pi}{2}+t\right)\right]$$

$$= \frac{1}{\pi}\left[\sin\left(\frac{\pi}{2}+t\right) + \sin\left(\frac{\pi}{2}-t\right)\right] = \frac{1}{\pi}\left[\cos t + \cos t\right]$$

$$= \frac{2\cos t}{\pi} \neq \text{constant}$$

Since $E[X(t)]$ is a func. of t, the random process $X(t)$ is not a stationary process.

② Consider the random process $X(t) = \cos(\omega_0 t + \theta)$, where θ is uniformly distributed in the interval $-\pi$ to π. Check whether $X(t)$ is stationary or not. Find the first & second moments of the process.

Sol: Since θ is uniformly distributed in $(-\pi, \pi)$, $f(\theta) = \frac{1}{2\pi}$, $-\pi < \theta < \pi$.

The first moment $E[X(t)] = E[\cos(\omega_0 t + \theta)] = \int_{-\pi}^{\pi} \cos(\omega_0 t + \theta) \frac{1}{2\pi} d\theta$

$$= \frac{1}{2\pi}\Big[\sin(\omega_0 t + \theta)\Big]_{-\pi}^{\pi}$$

$$= \frac{1}{2\pi}\left[\sin(\omega_0 t + \pi) - \sin(\omega_0 t - \pi)\right]$$

$$= \frac{1}{2\pi}\left[\sin(\omega_0 t + \pi) + \sin(\pi - \omega_0 t)\right]$$

$$= \frac{1}{2\pi}\left[-\sin\omega_0 t + \sin\omega_0 t\right] = 0 \quad \text{(constant)}$$

The second moment $E[X^2(t)] = E[\cos^2(\omega_0 t + \theta)]$

$$= E\left[\frac{1 + \cos 2(\omega_0 t + \theta)}{2}\right]$$

$$= \frac{1}{2}\left[E(1) + E(\cos(2\omega_0 t + 2\theta))\right]$$

$$= \frac{1}{2}\left[\int_{-\pi}^{\pi} \frac{1}{2\pi} d\theta + \int_{-\pi}^{\pi} \cos(2\omega_0 t + 2\theta) \frac{1}{2\pi} d\theta\right]$$

$$E[x^2(t)] = \frac{1}{4\pi}\left[(\theta)_{-\pi}^{\pi} + \left(\frac{\sin(2\omega_0 t + 2\theta)}{2} \right)_{-\pi}^{\pi} \right]$$

$$= \frac{1}{4\pi}\left[2\pi + \frac{1}{2}\left(\sin(2\omega_0 t + 2\pi) - \sin(2\omega_0 t - 2\pi) \right) \right]$$

$$= \frac{1}{4\pi}\left[2\pi + \frac{1}{2}\left(\sin 2\omega_0 t + \sin(2\pi - 2\omega_0 t) \right) \right]$$

$$= \frac{1}{4\pi}\left[2\pi + \frac{1}{2}\left(\sin 2\omega_0 t - \sin 2\omega_0 t \right) \right] = \frac{1}{2} \text{ (constant)}$$

$$\mathrm{Var}[X(t)] = E[X^2(t)] - [E(X(t))]^2$$

$$= \frac{1}{2} - (0)^2 = \frac{1}{2} \text{ (constant)}$$

∴ X(t) is a stationary process.

③ The process $\{X(t)\}$ whose probability distribution under certain conditions is given by, $P\{X(t) = n\} = \begin{cases} \frac{(at)^{n-1}}{(1+at)^{n+1}}, & n = 1, 2, \ldots \\ \frac{at}{1+at}, & n = 0 \end{cases}$. Show that it is not stationary (or evolutionary).

Sol: The probability distribution of X(t) is

$X(t) = n$	:	0	1	2	3	…
$P(X(t) = n) = P_n$	:	$\frac{at}{1+at}$	$\frac{1}{(1+at)^2}$	$\frac{at}{(1+at)^3}$	$\frac{(at)^2}{(1+at)^4}$	…

$$E(X(t)) = \sum_{n=0}^{\infty} n P_n = 0.\frac{at}{1+at} + \frac{1}{(1+at)^2} + \frac{2at}{(1+at)^3} + \frac{3(at)^2}{(1+at)^4} + \cdots$$

$$= \frac{1}{(1+at)^2}\left[1 + \frac{2at}{1+at} + 3\left(\frac{at}{1+at}\right)^2 + \cdots \right]$$

$$= \frac{1}{(1+at)^2}\left[1 - \frac{at}{1+at} \right]^{-2} \quad (\because (1-x)^{-2} = 1 + 2x + 3x^2 + \cdots)$$

$$= \frac{1}{(1+at)^2}(1+at)^2 = 1 \text{ (constant)}$$

$$E[x^2(t)] = \sum_{n=0}^{\infty} n^2 P_n = \cancel{\sum_{n=0}^{\infty} 0.\frac{at}{1+at} + 1^2.\frac{1}{(1+at)^2} + 2^2.\frac{at}{(1+at)^3} + \cdots}$$

$$= 0.\frac{at}{1+at} + \sum_{n=1}^{\infty} n^2.\frac{(at)^{n-1}}{(1+at)^{n+1}} = \frac{1}{(1+at)^2}\sum_{n=1}^{\infty} n^2.\frac{(at)^{n-1}}{(1+at)^{n-1}}$$

$$= \frac{1}{(1+at)^2}\sum_{n=1}^{\infty} (n^2 + n - n)\left(\frac{at}{1+at}\right)^{n-1}$$

$$= \frac{1}{(1+at)^2}\left[\sum_{n=1}^{\infty} n(n+1)\left(\frac{at}{1+at}\right)^{n-1} - \sum_{n=1}^{\infty} n\left(\frac{at}{1+at}\right)^{n-1} \right]$$

$$= \frac{1}{(1+at)^2}\left[\left(1.2 + 2.3\left(\frac{at}{1+at}\right) + 3.4\left(\frac{at}{1+at}\right)^2 + \cdots\right) - \left(1 + 2\left(\frac{at}{1+at}\right) + 3\left(\frac{at}{1+at}\right)^2 + \cdots\right) \right]$$

$$= \frac{1}{(1+at)^2}\left[2\left(1 + 3\left(\frac{at}{1+at}\right) + 6\left(\frac{at}{1+at}\right)^2 + \cdots\right) - \left(1 - \frac{at}{1+at}\right)^{-2} \right]$$

$$\bullet\ E[x^2(t)] = \frac{1}{(1+at)^2}\left[2\left(1-\frac{at}{1+at}\right)^{-3}-\left(1-\frac{at}{1+at}\right)^{-2}\right]$$

$$= \frac{1}{(1+at)^2}\left[2(1+at)^3-(1+at)^2\right] = 2(1+at)-1 = 2+2at-1$$

$$= 1+2at$$

$$Var[x(t)] = E[x^2(t)] - [E(x(t))]^2$$

$$= 1+2at-1 = 2at$$

Since Var[x(t)] is a func. of t, the given process is not stationary.

Wide Sense Stationary: (WSS) [Strict sense stationary of order 2 (or) Covariance stationary.

(i) $E[X(t)] =$ Constant (ii) $R(t_1,t_2) = E[X(t_1)X(t_2)] =$ a func. of (t_1-t_2).

Problems:

① Show that the random process $X(t) = A\cos(\omega t+\theta)$ is wide sense stationary if A & ω are constant & θ is uniformly distributed random variable in $(0,2\pi)$

Sol: Given $X(t) = A\cos(\omega t+\theta)$

Since θ is uniformly distributed in $(0,2\pi)$, $f(\theta) = \frac{1}{2\pi}$, $0<\theta<2\pi$.

$$E[X(t)] = E[A\cos(\omega t+\theta)] = A\int_0^{2\pi}\cos(\omega t+\theta).f(\theta)d\theta$$

$$= \frac{A}{2\pi}\left[\sin(\omega t+\theta)\right]_0^{2\pi} = \frac{A}{2\pi}\left[\sin(\omega t+2\pi)-\sin\omega t\right]$$

$$= \frac{A}{2\pi}\left[\sin\omega t - \sin\omega t\right] = 0 \text{ (constant)}$$

The auto correlation, $R(t_1,t_2) = E[X(t_1)X(t_2)]$

$$R(t_1,t_2) = E[A\cos(\omega t_1+\theta).A\cos(\omega t_2+\theta)]$$

$$= \frac{A^2}{2}E[\cos(\omega(t_1+t_2)+2\theta)+\cos(\omega(t_1-t_2))]$$

$$= \frac{A^2}{2}E[\cos(\omega(t_1+t_2)+2\theta)] + \frac{A^2}{2}\cos\omega(t_1-t_2)$$

$$= \frac{A^2}{2}\int_0^{2\pi}\cos(\omega(t_1+t_2)+2\theta)f(\theta)d\theta + \frac{A^2}{2}\cos\omega(t_1-t_2)$$

$$= \frac{A^2}{4\pi}\left[\frac{\sin(\omega(t_1+t_2)+2\theta)}{2}\right]_0^{2\pi} + \frac{A^2}{2}\cos\omega(t_1-t_2)$$

$$= \frac{A^2}{8\pi}\left[\sin(\omega(t_1+t_2)+4\pi)-\sin\omega(t_1+t_2)\right] + \frac{A^2}{2}\cos\omega(t_1-t_2)$$

$$= \frac{A^2}{8\pi}\left[\sin\omega(t_1+t_2)-\sin\omega(t_1+t_2)\right] + \frac{A^2}{2}\cos\omega(t_1-t_2)$$

$$= \frac{A^2}{2}\cos\omega(t_1-t_2) = \text{a func. of } (t_1-t_2)$$

$\therefore X(t)$ is a wide sense stationary process.

② Given a random variable Y with characteristic fun. $\phi(\omega) = E[e^{i\omega y}]$ & random process defined by $X(t) = \cos[\lambda t + y]$, show that $[X(t)]$ is stationary in the wide sense of $\phi(1) = \phi(2) = 0$.

Sol: Given that $\phi(1) = \phi(2) = 0$

Now consider $E[X(t)] = E[\cos(\lambda t + y)]$

$= E[\cos\lambda t \cos y - \sin\lambda t \sin y]$

$= \cos\lambda t \, E[\cos y] - \sin\lambda t \, E[\sin y]$ —①

$\cos(A-B) = \cos A\cos B + \sin A \sin B$
$\cos(A+B) = \cos A \cos B - \sin A \sin B$
$\sin(A-B) = \sin A \cos B - \cos A \sin B$
$\sin(A+B) = \sin A \cos B + \cos A \sin B$

$\because \phi(1) = 0 \Rightarrow E[e^{iy}] = 0 \Rightarrow E[\cos y + i \sin y] = 0$

$\Rightarrow E[\cos y] + i E[\sin y] = 0$

$\Rightarrow E[\cos y] = 0 = E[\sin y]$

Similarly, $\phi(2) = 0 \Rightarrow E[e^{i2y}] = 0 \Rightarrow E[\cos 2y] + i E[\sin 2y] = 0$

$\Rightarrow E[\cos 2y] = 0 = E[\sin 2y]$

$\therefore$ ① $\Rightarrow E[X(t)] = 0$ (constant)

$R(t_1, t_2) = E[X(t_1) . X(t_2)] = E[\cos(\lambda t_1 + y) . \cos(\lambda t_2 + y)]$

$= E[(\cos\lambda t_1 \cos y - \sin\lambda t_1 \sin y) . (\cos\lambda t_2 \cos y - \sin\lambda t_2 \sin y)]$

$= E[\cos\lambda t_1 \cos\lambda t_2 \cos^2 y - \sin\lambda t_1 \cos\lambda t_2 \sin y \cos y - \sin\lambda t_2 \cos\lambda t_1 \sin y \cos y$
$+ \sin\lambda t_1 \sin\lambda t_2 \sin^2 y]$

$= \cos\lambda t_1 \cos\lambda t_2 E[\cos^2 y] - \sin\lambda t_1 \cos\lambda t_2 E\left[\frac{\sin 2y}{2}\right]$
$- \sin\lambda t_2 \cos\lambda t_1 E\left[\frac{\sin 2y}{2}\right] + \sin\lambda t_1 \sin\lambda t_2 E[\sin^2 y]$

$= \cos\lambda t_1 \cos\lambda t_2 E\left[\frac{1+\cos 2y}{2}\right] + \sin\lambda t_1 \sin\lambda t_2 E\left[\frac{1-\cos 2y}{2}\right]$ $(\because E(\sin 2y) = 0)$

$= \frac{1}{2}[\cos\lambda t_1 \cos\lambda t_2 + \sin\lambda t_1 \sin\lambda t_2]$ $(\because E(\cos 2y) = 0)$

$= \frac{1}{2}\cos\lambda(t_1 - t_2)$ = a fun. of $(t_1 - t_2)$

Hence $X(t)$ is a WSS process.

③ Two random process $X(t)$ & $Y(t)$ are defined by $X(t) = A\cos\omega t + B\sin\omega t$ & $Y(t) = B\cos\omega t - A\sin\omega t$ Show that $X(t)$ & $Y(t)$ are jointly wide-sense stationary if A & B are uncorrelated rvs with zero means & the same variances & ω is constant.

$2\sin A\cos B = \sin(A+B) + \sin(A-B)$
$2\cos A \sin B = \sin(A+B) - \sin(A-B)$
$2\cos A \cos B = \cos(A+B) + \cos(A-B)$
$2\sin A \sin B = \cos(A-B) - \cos(A+B)$

Sol: $X(t)$ & $Y(t)$ are jointly WSS if

(i) $E[X(t)]$ is WSS & $E[Y(t)]$ is WSS

(ii) $R_{XY}(t_1, t_2)$ is a fun. of $t_1 - t_2$.

- Given $E(A) = E(B) = 0$

$Var(A) = Var(B) \Rightarrow E(A^2) - [E(A)]^2 = E(B^2) - [E(B)]^2$

$\Rightarrow E(A^2) = E(B^2) \quad (\because E(A) = E(B) = 0)$

$E(A^2) = E(B^2) = k$ (Say)

Since A & B are uncorrelated, $Cov(A,B) = 0 \Rightarrow E(AB) - E(A)E(B) = 0$

$\Rightarrow E(AB) = 0 \quad (\because E(A) = E(B) = 0)$

First let us prove $X(t)$ & $Y(t)$ are individually WSS processes.

~~(i) E(X(t~~ To prove: $X(t)$ is a WSS, the following conditions are to be verified.

(i) $E[x(t)]$ must be a constant.

(ii) $R(t_1, t_2)$ is a funl. of $(t_1 - t_2)$.

Now $E[X(t)] = E[A\cos\omega t + B\sin\omega t]$

$= \cos\omega t \, E(A) + \sin\omega t \, E(B) = 0$ (constant)

& $R(t_1, t_2) = E[x(t_1) . x(t_2)]$

$= E[(A\cos\omega t_1 + B\sin\omega t_1).(A\cos\omega t_2 + B\sin\omega t_2)]$

$= E[A^2\cos\omega t_1 \cos\omega t_2 + AB\cos\omega t_1 \sin\omega t_2 + AB\cos\omega t_2 \sin\omega t_1 + B^2\sin\omega t_1 \sin\omega t_2]$

$= \cos\omega t_1 \cos\omega t_2 \, E(A^2) + \cos\omega t_1 \sin\omega t_2 \, E(AB) + \cos\omega t_2 \sin\omega t_1 \, E(AB) + \sin\omega t_1 \sin\omega t_2 \, E(B^2)$

$= k[\cos\omega t_1 \cos\omega t_2 + \sin\omega t_1 \sin\omega t_2] \quad (\because E(AB) = 0, E(A^2) = E(B^2) = k)$

$= k\cos(\omega t_1 - \omega t_2) = k\cos\omega(t_1 - t_2)$

$=$ a funl. of $(t_1 - t_2)$.

Hence $X(t)$ is a WSS process.

Similarly we can prove that $Y(t)$ is a WSS process.

To prove $X(t)$ & $Y(t)$ are jointly WSS process.

Consider, $R_{XY}(t_1, t_2) = E[X(t_1) . Y(t_2)]$

$= E[(A\cos\omega t_1 + B\sin\omega t_1).(B\cos\omega t_2 - A\sin\omega t_2)]$

$= E[AB\cos\omega t_1 \cos\omega t_2 - A^2\cos\omega t_1 \sin\omega t_2 + B^2\sin\omega t_1 \cos\omega t_2 - AB\sin\omega t_1 \sin\omega t_2]$

$= E(AB)[\cos\omega t_1 \cos\omega t_2 - \sin\omega t_1 \sin\omega t_2] - E(A^2)\cos\omega t_1 \sin\omega t_2 + E(B^2)\sin\omega t_1 \cos\omega t_2$

$= k[\sin\omega t_1 \cos\omega t_2 - \cos\omega t_1 \sin\omega t_2] \quad (\because E(AB) = 0)$

⑤ If [X(t)] is a WSS process with auto correlation funl. $R_{XX}(\tau)$ & if $Y(t)= X(t+a) - X(t-a)$. Show that $R_{YY}(\tau) = 2R_{XX}(\tau) - R_{XX}(\tau+2a)$
$- R_{XX}(\tau-2a)$.

Sol: By definition, $R_{XX}(\tau)=E[X(t)X(t+\tau)]$ — ①

Given $Y(t)= X(t+a) - X(t-a)$ — ②

Consider, LHS $R_{YY}(\tau) = E[Y(t)Y(t+\tau)]$ — ③

Subst. ② in ③, we get

$$R_{YY}(\tau) = E[(X(t+a)-X(t-a))(X(t+a+\tau)-X(t-a+\tau))]$$
$$= E[X(t+a)X(t+a+\tau) - X(t-a)X(t+a+\tau)$$
$$- X(t+a)X(t-a+\tau) + X(t-a)X(t-a+\tau)]$$
$$= E[X(t+a)X(t+a+\tau)] - E[X(t-a)X(t-a+(2a+\tau))]$$
$$- E[X(t+a)X(t+a+(\tau-2a))] + E[X(t-a)X(t-a+\tau)]$$

$[\because t+a+\tau = t-a+2a+\tau$
$t-a+\tau = t+a-2a+\tau]$

Let $z=t+a$; $u=t-a$

Then $R_{YY}(\tau) = E[X(z)X(z+\tau)] - E[X(u)X(u+2a+\tau)]$
$- E[X(z)X(z+\tau-2a)] + E[X(u)X(u+\tau)]$
$= R_{XX}(\tau) - R_{XX}(2a+\tau) - R_{XX}(\tau-2a) + R_{XX}(\tau)$ (By ①)
$= 2R_{XX}(\tau) - R_{XX}(\tau+2a) - R_{XX}(\tau-2a)$
= R.H.S.

---

⑥ For a random process $X(t)=Y\sin\omega t$, Y is an uniform random variable in the interval -1 to 1. Check whether the process is wide sense stationary or not.

Sol: Y is uniformly distributed in the interval -1 to 1.

$X(t) = Y\sin\omega t$, $f(y) = \frac{1}{2}$, $-1 \le y < 1$

Mean $= E[X(t_0)] = \int_{-1}^{1} y\sin\omega t \cdot \frac{1}{2}\, dy = \frac{1}{2}\sin\omega t \int_{-1}^{1} y\,dy$
$= \frac{1}{2}\sin\omega t \left[\frac{y^2}{2}\right]_{-1}^{1} = \frac{1}{4}\sin\omega t[1-1] = 0$

$E[X(t_1)X(t_2)] = E[Y\sin\omega t_1 \cdot Y\sin\omega t_2]$
$= \sin\omega t_1 \sin\omega t_2\, E[Y^2]$
$= \sin\omega t_1 \sin\omega t_2 \int_{-1}^{1} y^2 \cdot \frac{1}{2}\, dy = \frac{1}{2}\sin\omega t_1 \sin\omega t_2 \left[\frac{y^3}{3}\right]_{-1}^{1}$

$= \frac{1}{6} \sin\omega t_1 \sin\omega t_2 \,(1-(-1))$

$= \frac{1}{3} \sin\omega t_1 \sin\omega t_2 = \frac{1}{6}\left[\cos\omega(t_1-t_2) - \cos\omega(t_1+t_2)\right]$

Mean is constant, the auto correlation func. is not a func. of time differences of the two random variables.

Hence the process is not wide sense stationary.

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⑦ If $x(t) = Y\cos t + Z\sin t$ for all t where Y & Z are independent binary random variables; each of which assumes the values -1 & 2 with probabilities $\frac{2}{3}$ & $\frac{1}{3}$ respectively, prove that $\{x(t)\}$ is wide sense stationary.

Sol: Y & Z are discrete random variable which assumes values

Y	-1	2
$P(Y)$	$2/3$	$1/3$

Z	-1	2
$P(Z)$	$2/3$	$1/3$

$E(Y) = \sum y_i P(y_i) = (-1)\left(\frac{2}{3}\right) + (2)\left(\frac{1}{3}\right) = \frac{-2}{3} + \frac{2}{3} = 0$

$E(Y^2) = \sum y_i^2 P(y_i) = (-1)^2\left(\frac{2}{3}\right) + (2)^2\left(\frac{1}{3}\right) = \frac{2}{3} + \frac{4}{3} = 2$

$E(Z) = \sum z_i P(z_i) = (-1)(2/3) + 2(1/3) = 0$

$E(Z^2) = \sum z_i^2 P(z_i) = (-1)^2(2/3) + (2)^2(1/3) = 2$

Y & Z are independent random variables

$\therefore E(YZ) = E(Y).E(Z) = 0$

(i) $E[x(t)] = E[Y\cos t + Z\sin t] = E(Y)\cos t + E(Z)\sin t = 0 = \text{constant}$

(ii) $R_{XX}(t_1, t_2) = E[x(t_1).x(t_2)]$

$= E[(Y\cos t_1 + Z\sin t_1)(Y\cos t_2 + Z\sin t_2)]$

$= E[Y^2\cos t_1\cos t_2 + YZ\cos t_1\sin t_2 + ZY\sin t_1\cos t_2 + Z^2\sin t_1\sin t_2]$

$= E(Y^2)\cos t_1\cos t_2 + E(YZ)\cos t_1\sin t_2 + E(ZY)\sin t_1\cos t_2 + E(Z^2)\sin t_1\sin t_2$

$= 2\cos t_1\cos t_2 + 2\sin t_1\sin t_2 \quad (\because E(YZ) = E(ZY) = 0)$

$= 2(\cos t_1\cos t_2 + \sin t_1\sin t_2) = 2\cos(t_1-t_2)$

Hence by (i) & (ii), $x(t)$ is covariance stationary. (ie) WSS process.

Markov Processes:

Markov process is one in which the future value is independent of the past values, given the present value.

Markovian:

A random process $\{X(t)\}$ is said to be Markovian if

$$P[X(t_{n+1}) = x_{n+1} / X(t_n) = x_n , X(t_{n-1}) = x_{n-1}, \dots, X(t_0) = x_0]$$
$$= P[X(t_{n+1}) = x_{n+1} / X(t_n) = x_n]$$

where $t_0 \le t_1 \le t_2 \le \dots \le t_n \le t_{n+1}$. Here $X_0, X_1, X_2, \dots, X_n, X_{n+1}$ are called the states of the process. If the random process at time t_n is in the state X_n, the future state of the random process X_{n+1} at t_{n+1} depends only on the present state X_n & not on the past states $X_{n-1}, X_{n-2}, \dots, X_0$.

Examples of Markov process:

- * The probability of raining today depends only on previous weather conditions existed for the last two days & not on past weather conditions.
- * A difference eqn/. is Markovian.

Classification of Markov Process:

A Markov process can be classified into four types, based on the values taken by time t & the state space $\{x_i\}$.

(i) A continuous random process satisfying Markov property is known as continuous parameter Markov process. Note that for a continuous random process both t & $\{x_i\}$ are continuous.

(ii) A continuous random sequence satisfying Markov property is known as discrete parameter Markov process as the parameter t is discrete but $\{x_i\}$ is continuous.

(iii) A discrete random sequence satisfying Markov property is known as the discrete parameter Markov chain as t is discrete & $\{x_i\}$ is also discrete.

(iv) A discrete random process satisfying Markov property is called as continuous parameter Markov chain because the parameter t is continuous & $\{x_i\}$ is discrete.

Markov Chain:

Let $\{X(t)\}$ be a Markov process which possess Markov property &"
which takes only discrete values whether t is discrete or continuous. Then
$\{X(t)\}$ is called as Markov chain. We define the Markov chain as follows.
If $P\{X_n = a_n / X_{n-1} = a_{n-1}, X_{n-2} = a_{n-2}, \dots, X_0 = a_0\}$
$\Rightarrow P\{X_n = a_n / X_{n-1} = a_{n-1}\}$ for all n, then the process X_n; $n = 0, 1, 2, \dots$
called as Markov chain. Here $a_1, a_2, \dots, a_n$ are called the states of the
Markov chain.

One-Step Transition Probability:

The conditional probability $P\{X_n = a_j / X_{n-1} = a_i\}$ is called the one-step transition probability from state a_i to state a_j at the n^{th} step (trial) & is denoted by $P_{ij}(n-1, n)$.

The transition probability $P_{ij} = P(X_{t+1} = j / X_t = i)$ does not depend on the time t, we call the process homogeneous.

Homogeneous Markov Chain:

If $P_{ij}(n-1, n) = P_{ij}(m-1, m)$, then the Markov chain is called the homogeneous Markov chain or the chain is said to have stationary transition probabilities.

Transition Probability Matrix:

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When the Markov chain is homogeneous, the one-step transition probability is denoted by P_{ij}. The matrix $P = \{P_{ij}\}$ is called (one-step) transition probability matrix (tpm).

The tpm of a chain is a Stochastic matrix, since all $P_{ij} \geq 0$ & the sum of all elements of any row of the transition matrix is equal to 1.

n-Step Transition Probability:

The conditional probability that the process is in state a_j at step n, given that it was in state a_i at step 0. (ie) $P\{X_n = a_j / X_0 = a_i\}$ is called the n-step transition probability & denoted by $P_{ij}^{(n)}$.

Chapman-Kolmogorov theorem:

If P is the transition probability matrix of a homogeneous Markov chain, then the n-step tpm $P^{(n)}$ is equal to P^n. (ie) $[P_{ij}^{(n)}] = [P_{ij}]^n$.

Regular Matrix:

A stochastic matrix P is said to be a regular matrix, if all the entries of P^m (for some +ve integer m) are positive.

If the tpm is regular, then we say that the homogeneous Markov chain is regular.

Classification of states of a Markov chain:

Irreducible: A Markov chain is said to be irreducible if every state can be reached from every other state, where $P_{ij}^{(n)} > 0$ for some n & for all i & j. The tpm of an irreducible chain is an irreducible matrix. Otherwise the chain is said to be non-irreducible or reducible.

Return State: If $P_{ii}^{(n)} > 0$, for some $n > 1$, then we call the state i of the Markov chain as return state.

Period: Let $P_{ii}^{(m)} > 0$ for all m. Let i be a return state. Then we define the period d_i as follows. $d_i = GCD\{m : P_{ij}^{(m)} > 0\}$, where GCD stands for the greatest common divisor

If $d_i = 1$, then state i is said to be aperiodic. If $d_i > 1$, then state i is said to be periodic. Obviously state i is periodic if $P_{ii} \neq 0$.

Recurrence Time Probability: The probability that the chain returns to state i, starting from state i, for the first time at the n^{th} step is called the recurrence time probability or the first return time probability. & is denoted by $f_{ii}^{(n)}$. $\{n, f_{ii}^{(n)}\}$, $n = 1, 2, 3, \ldots$ is the distribution of recurrence times of the state i.

If $F_{ii} = \sum_{n=1}^{\infty} f_{ii}^{(n)} = 1$, the return to state i is certain.

$\mu_{ii} = \sum_{n=1}^{\infty} n f_{ii}^{(n)}$ is called the mean recurrence time of the state i.

A state i is said to be persistent or recurrent of the return state i is certain, (ie) if $F_{ii} = 1$. The state i is said to be transient if the return state i is uncertain, (ie) if $F_{ii} < 1$. The state i is said to be non-null persistent if its mean recurrence time μ_{ii} is finite & null persistent if $\mu_{ii} = \infty$.

A non-null persistent & aperiodic state is called ergodic.

Result:

* If a Markov chain is irreducible, all its states are of the same type. They are all transient, all null persistent or all non-null persistent. All its states are either aperiodic or periodic with the same period.

* If a Markov chain is finite irreducible, all its states are non-null persistent.

Problems:

① The tpm of the Markov chain $\{X_n\}$, $n=1,2,3,\dots$ having 3 states 1, 2 & 3 is

$$P = \begin{array}{c|ccc} & 1 & 2 & 3 \\ 1 & 0.1 & 0.5 & 0.4 \\ 2 & 0.6 & 0.2 & 0.2 \\ 3 & 0.3 & 0.4 & 0.3 \end{array}$$

& the initial distribution is $p^{(0)} = (0.7, 0.2, 0.1)$.

Find (i) $P(X_2=3)$ & (ii) $P(X_3=2, X_2=3, X_1=3, X_0=2)$.

Sol:

$$p^{(2)} = P^2 = \begin{pmatrix} 0.1 & 0.5 & 0.4 \\ 0.6 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.3 \end{pmatrix}\begin{pmatrix} 0.1 & 0.5 & 0.4 \\ 0.6 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.3 \end{pmatrix}$$

$$= \begin{pmatrix} 0.43 & 0.31 & 0.26 \\ 0.24 & 0.42 & 0.34 \\ 0.36 & 0.35 & 0.29 \end{pmatrix}$$

(i) $P(X_2=3) = P(X_2=3, X_0=1) + P(X_2=3, X_0=2) + P(X_2=3, X_0=3)$

$= P(X_2=3/X_0=1).P(X_0=1) + P(X_2=3/X_0=2).P(X_0=2) + P(X_2=3/X_0=3).P(X_0=3)$

$= P_{13}^{(2)}.P(X_0=1) + P_{23}^{(2)}.P(X_0=2) + P_{33}^{(2)}.P(X_0=3)$

$= (0.26 \times 0.7) + (0.34 \times 0.2) + (0.29 \times 0.1)$

$= 0.279$

(ii) $P[X_3=2, X_2=3, X_1=3, X_0=2] = P[X_3=2/X_2=3, X_1=3, X_0=2]. P[X_2=3, X_1=3, X_0=2]$

$= P[X_3=2/X_2=3].P[X_2=3/X_1=3, X_0=2].P[X_1=3, X_0=2]$

$= P_{32}^{(1)}.P[X_2=3/X_1=3].P[X_1=3/X_0=2].P[X_0=2]$

$= P_{32}^{(1)}.P_{33}^{(1)}.P_{23}^{(1)}.P[X_0=2]$

$= (0.4)(0.3)(0.2)(0.2)$

$= 0.0048$

② The tpm of a Markov chain with 3 states 0,1,2 is $P = \begin{pmatrix} 3/4 & 1/4 & 0 \\ 1/4 & 1/2 & 1/4 \\ 0 & 3/4 & 1/4 \end{pmatrix}$ & the initial state distribution of the chain is $P(X_0 = i) = 1/3$, $i = 0,1,2$. Find (i) $P(X_2 = 2)$ & (ii) $P(X_3 = 1, X_2 = 2, X_1 = 1, X_0 = 2)$.

Sol: Given $P = \begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & 3/4 & 1/4 & 0 \\ 1 & 1/4 & 1/2 & 1/4 \\ 2 & 0 & 3/4 & 1/4 \end{array}$

$$P^2 = \begin{pmatrix} 3/4 & 1/4 & 0 \\ 1/4 & 1/2 & 1/4 \\ 0 & 3/4 & 1/4 \end{pmatrix}\begin{pmatrix} 3/4 & 1/4 & 0 \\ 1/4 & 1/2 & 1/4 \\ 0 & 3/4 & 1/4 \end{pmatrix} = \begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & 5/8 & 5/16 & 1/16 \\ 1 & 5/16 & 1/2 & 3/16 \\ 2 & 3/16 & 9/16 & 1/4 \end{array}$$

(i) $P(X_2 = 2) = P(X_2 = 2, X_0 = 0) + P(X_2 = 2, X_0 = 1) + P(X_2 = 2, X_0 = 2)$

$= P(X_2 = 2 / X_0 = 0) . P(X_0 = 0) + P(X_2 = 2 / X_0 = 1) . P(X_0 = 1) + P(X_2 = 2 / X_0 = 2) . P(X_0 = 2)$

$= P_{02}^{(2)} . P(X_0 = 0) + P_{12}^{(2)} . P(X_0 = 1) + P_{22}^{(2)} . P(X_0 = 2)$

$= \left(\frac{1}{16}\right)\left(\frac{1}{3}\right) + \left(\frac{3}{16}\right)\left(\frac{1}{3}\right) + \left(\frac{1}{4}\right)\left(\frac{1}{3}\right)$

$= \frac{1}{3}\left[\frac{1}{16} + \frac{3}{16} + \frac{1}{4}\right] = \frac{1}{3} \cdot \frac{8}{16} = \frac{1}{6}$

(ii) $P(X_3 = 1, X_2 = 2, X_1 = 1, X_0 = 2) = P(X_3 = 1 / X_2 = 2, X_1 = 1, X_0 = 2) . P(X_2 = 2, X_1 = 1, X_0 = 2)$

$= P(X_3 = 1 / X_2 = 2) . P(X_2 = 2 / X_1 = 1, X_0 = 2) . P(X_1 = 1, X_0 = 2)$

~~$\neq P_{12}$~~ $= P_{21}^{(1)} . P(X_2 = 2 / X_1 = 1) . P(X_1 = 1 / X_0 = 2) . P(X_0 = 2)$

$= P_{21}^{(1)} . P_{12}^{(1)} . P_{21}^{(1)} . P(X_0 = 2)$

$= \frac{3}{4} \cdot \frac{1}{4} \cdot \frac{3}{4} \cdot \frac{1}{3} = \frac{3}{64}$

③ Prove that the matrix $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{pmatrix}$ is the tpm of an irreducible Markov chain.

Sol: Given $P = \begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & 0 & 1 & 0 \\ 2 & 0 & 0 & 1 \\ 3 & 1/2 & 1/2 & 0 \end{array}$

$$P^2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \end{pmatrix}$$

$$P^3 = P^2 . P = \begin{pmatrix} 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{pmatrix} = \begin{pmatrix} 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \\ 1/4 & 1/4 & 1/2 \end{pmatrix}$$

Here $P_{11}^{(3)} = \frac{1}{2} > 0$, $P_{12}^{(1)} = 1 > 0$, $P_{13}^{(2)} = 1 > 0$

$P_{21}^{(2)} = \frac{1}{2} > 0$, $P_{22}^{(2)} = \frac{1}{2} > 0$, $P_{23}^{(1)} = 1 > 0$

$P_{31}^{(1)} = \frac{1}{2} > 0$, $P_{32}^{(1)} = \frac{1}{2} > 0$, $P_{33}^{(2)} = \frac{1}{2} > 0$

$\therefore P_{ij}^{(n)} > 0$ for all i & j & some n.

Hence the ~~tpm~~ matrix of the tpm is irreducible.

(4) Three boys A, B, C are throwing a ball each other. A always throws the ball to B & B always throws the ball to C, but C is just as likely to throw the ball to B as to A. ~~S.T. the process is Markovian~~. Find the transition matrix & classify the states.

Sol: Transition probability matrix:

$$\text{States of } X_{n-1} \begin{array}{c|ccc} & \multicolumn{3}{c}{\text{States of } X_n} \\ & A & B & C \\ \hline A & 0 & 1 & 0 \\ B & 0 & 0 & 1 \\ C & \frac{1}{2} & \frac{1}{2} & 0 \end{array}$$

$$P^2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

$$P^3 = \begin{pmatrix} 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{pmatrix}$$

$$P^4 = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \end{pmatrix}$$

$$P^5 = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{8} & \frac{3}{8} & \frac{1}{2} \end{pmatrix} \quad \ldots \text{ \& so on.}$$

Here $P_{11}^{(3)}, P_{11}^{(5)}, \ldots$ are > 0 and $\gcd\{3, 5, \ldots\} = 1 = d_1$

× [$P_{22}^{(2)}, P_{22}^{(3)}, P_{22}^{(4)}, P_{22}^{(5)}, \ldots$ are > 0 and $d_2 = \gcd\{2, 3, 4, 5, \ldots\} = 1$

$P_{33}^{(2)}, P_{33}^{(3)}, P_{33}^{(4)}, P_{33}^{(5)}, \ldots$ are > 0 and $d_3 = \gcd\{2, 3, 4, 5, \ldots\} = 1$] ×

$\therefore$ States A, B & C are apriodic, (i.e) periodic with period 1.

Here, $P_{11}^{(3)} = \frac{1}{2} > 0$, $P_{12}^{(1)} = 1 > 0$, $P_{13}^{(2)} = 1 > 0$

$P_{21}^{(2)} = \frac{1}{2} > 0$, $P_{22}^{(2)} = \frac{1}{2} > 0$, $P_{23}^{(1)} = 1 > 0$

$P_{31}^{(1)} = \frac{1}{2} > 0$, $P_{32}^{(1)} = \frac{1}{2} > 0$, $P_{33}^{(2)} = \frac{1}{2} > 0$

Hence the chain is irreducible.

Since the chain is finite & irreducible, all its states are non-null persistent.

Since the chain is non-null persistent & aperiodic, all its states are ergodic.

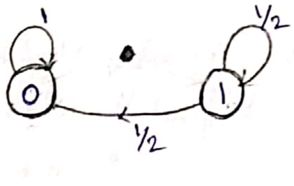
---

⑤ Consider a Markov chain with state space $\{0,1\}$ & the tpm $P = \begin{array}{c} \\ 0 \\ 1 \end{array}\begin{array}{cc} 0 & 1 \\ \end{array}$ $P = \begin{array}{c|cc} & 0 & 1 \\ \hline 0 & 1 & 0 \\ 1 & \frac{1}{2} & \frac{1}{2} \end{array}$

(i) Draw a transition diagram. (ii) S.T. state 0 is recurrent. (iii) S.T. state 1 is transient. (iv) Is the state 1 periodic? If so, what is the period? (v) Is the chain irreducible? (vi) Is the chain ergodic? Explain.

Sol:

(i) Transition diagram:



Given $P = \begin{pmatrix} 1 & 0 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$

$$P^2 = \begin{pmatrix} 1 & 0 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{3}{4} & \frac{1}{4} \end{pmatrix}$$

$$P^3 = \begin{pmatrix} 1 & 0 \\ \frac{3}{4} & \frac{1}{4} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{7}{8} & \frac{1}{8} \end{pmatrix}$$

$$P^4 = \begin{pmatrix} 1 & 0 \\ \frac{7}{8} & \frac{1}{8} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{15}{16} & \frac{1}{16} \end{pmatrix} \quad \ldots \text{ \& so on.}$$

(ii) State 0:

$f_{00}^{(1)} = 1, \quad f_{00}^{(2)} = 0, \quad f_{00}^{(3)} = 0, \ldots$

$\therefore F_{ii} = \sum_{n=1}^{\infty} f_{ii}^{(n)} \Rightarrow F_{00} = \sum_{n=1}^{\infty} f_{00}^{(n)} = 1 + 0 + 0 + \cdots = 1$ $\quad \therefore$ State 0 is recurrent.

(iii) State 1:

$f_{11}^{(1)} = \frac{1}{2}, \quad f_{11}^{(2)} = 0, \quad f_{11}^{(3)} = 0, \ldots$ $\quad \therefore$ State 1 is transient.

$\therefore F_{11} = \sum_{n=1}^{\infty} f_{11}^{(n)} = \frac{1}{2} + 0 + 0 + \cdots = \frac{1}{2} < 1$

(iv) [$P_{00}^{(1)}, P_{00}^{(2)}, P_{00}^{(3)}, P_{00}^{(4)}, \ldots$ are > 0 & $\gcd\{1,2,3,4,\ldots\} = 1 = d_0$]

$P_{11}^{(1)}, P_{11}^{(2)}, P_{11}^{(3)}, P_{11}^{(4)}, \ldots$ are > 0 & $d_1 = \gcd\{1,2,3,4,\ldots\} = 1$

$\therefore$ State 1 is periodic with period 1 (ie) aperiodic.

(v) $P_{00}^{(1)} = 1 > 0, \quad P_{01}^{(n)} = 0, \; n = 1,2,3,\ldots$

$P_{10}^{(1)} = \frac{1}{2} > 0, \quad P_{11}^{(1)} = \frac{1}{2} > 0$. Here $P_{01}^{(n)} = 0, n = 1,2,3,\ldots$

Hence the chain is not irreducible.

(vi) Since the chain is not irreducible, all the states are not non-null persistent. Hence the chain is not ergodic.

Defn.:

(A state is said to be an absorbing state if no other state is accessible from it.) (ii) for an absorbing state i, $P_{ii}=1$.

(A Markov chain is absorbing if,

(i) it has atleast one absorbing state

(ii) it is possible to go from every non-absorbing state to atleast one absorbing state (not necessarily in one step))

---

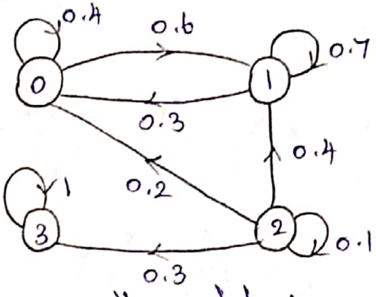
⑥ Consider the Markov chain with tpm

[Result:- Two states that communicate are in the same class.]

$$\begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & 0.4 & 0.6 & 0 & 0 \\ 1 & 0.3 & 0.7 & 0 & 0 \\ 2 & 0.2 & 0.4 & 0.1 & 0.3 \\ 3 & 0 & 0 & 0 & 1 \end{array}$$

Is it irreducible? If not find the classes. Find the nature of the states.

Sol: Transition diagram:



From the transition diagram, no other state is accessible from state 3 & $P_{33}^{(1)}=1$. ∴ State 3 is an absorbing state.

Since every state is not reachable from every other state, the Markov chain is not irreducible & so not ergodic. The classes of the chain are $\{0,1\},\{2\},\{3\}$.

A state $x \in \Omega$ is called essential if for all $y \ni: x \to y$ it is also true that $y \to x$

Defn.: A state i is called an essential state if it communicates with every state it leads to. (Assume ~~$P_{jk}(n)>0$~~ $P_{jk}(n)>0$ then if $P_{kj}(m)>0$, j is an essential state.

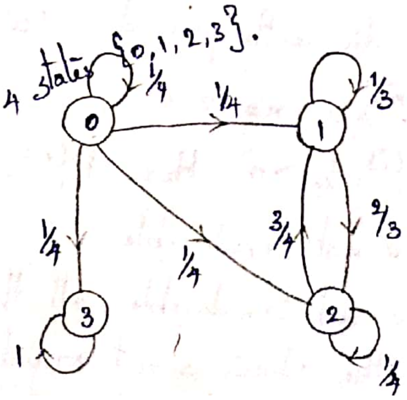
Irreducible <=> all the states are essential states

⑦ Assume a Markov chain with transition probability matrix

$$\begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ 1 & 0 & \frac{1}{3} & \frac{2}{3} & 0 \\ 2 & 0 & \frac{3}{4} & \frac{1}{4} & 0 \\ 3 & 0 & 0 & 0 & 1 \end{array}$$

. Find the essential states.

Sol: The given chain $\{X_n\}$ has 4 states $\{0,1,2,3\}$. Transition diagram:



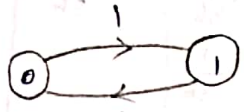
Every state is accessible from 0 but not 0 from others. So, 0 is not an essential state. States 1 & 2 communicate with each other, but not with others. Hence states 1 & 2 are not essential states. State 3 is not communicate with other states. ∴ State 3 is not an essential state.

Hence there is no essential states in the chain.

⑧ Consider a Markov chain with state $\{0,1\}$ & tpm $P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Is the state 0 periodic? If so what is the period?

Sol: Given $P = \begin{array}{c|cc} & 0 & 1 \\ \hline 0 & 0 & 1 \\ 1 & 1 & 0 \end{array}$

Transition diagram:



$P_{00}^{(1)} = 0$, $P_{00}^{(2)} = 1.1 = 1 > 0$, $P_{00}^{(3)} = 0$, $P_{00}^{(4)} = \overset{1.1.1.1 = 1 > 0}{\bullet}$, . . .

$d_i = GCD\{m : P_{ii}^{(m)} > 0\}$

$d_0 = GCD\{2, \overset{4, \ldots}{\wedge}\} = 2$

∴ State 0 is periodic with period 2.

⑨ A fair die is tossed repeatedly. If X_n denotes the maximum of the numbers occurring in the first n tosses, find the tpm P of the Markov chain $\{X_n\}$. Find also P^2 & $P[X_2 = 6]$.

Sol: The state space is given by $\{1,2,3,4,5,6\}$. The tpm is formed by using the following analysis.

Let X_n = Maximum of the numbers occurring in the first n trials 3, say.

Then $X_{n+1} = \begin{cases} 3 & \text{, if } (n+1)^{th} \text{ trial results in } 1, 2 \text{ or } 3 \\ 4 & \text{, if } (n+1)^{th} \text{ trial results in } 4 \\ 5 & \text{, if } (n+1)^{th} \text{ trial results in } 5 \\ 6 & \text{, if } (n+1)^{th} \text{ trial results in } 6 \end{cases}$

$\therefore P\{X_{n+1} = 3 / X_n = 3\} = \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{3}{6}$

For $i = 4, 5, 6$, $P\{X_{n+1} = i / X_n = 3\} = \frac{1}{6}$

The tpm of the chain is given by

$$P = X_n \begin{array}{c} \\ \end{array} \overset{X_{n+1}}{\begin{array}{c|cccccc} & 1 & 2 & 3 & 4 & 5 & 6 \\ \hline 1 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ 2 & 0 & \frac{2}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ 3 & 0 & 0 & \frac{3}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ 4 & 0 & 0 & 0 & \frac{4}{6} & \frac{1}{6} & \frac{1}{6} \\ 5 & 0 & 0 & 0 & 0 & \frac{5}{6} & \frac{1}{6} \\ 6 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}}$$

$$P^2 = \begin{bmatrix} \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ 0 & \frac{2}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ 0 & 0 & \frac{3}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ 0 & 0 & 0 & \frac{4}{6} & \frac{1}{6} & \frac{1}{6} \\ 0 & 0 & 0 & 0 & \frac{5}{6} & \frac{1}{6} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ 0 & \frac{2}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ 0 & 0 & \frac{3}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ 0 & 0 & 0 & \frac{4}{6} & \frac{1}{6} & \frac{1}{6} \\ 0 & 0 & 0 & 0 & \frac{5}{6} & \frac{1}{6} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{36} & \frac{3}{36} & \frac{5}{36} & \frac{7}{36} & \frac{9}{36} & \frac{11}{36} \\ 0 & \frac{4}{36} & \frac{5}{36} & \frac{7}{36} & \frac{9}{36} & \frac{11}{36} \\ 0 & 0 & \frac{9}{36} & \frac{7}{36} & \frac{9}{36} & \frac{11}{36} \\ 0 & 0 & 0 & \frac{16}{36} & \frac{9}{36} & \frac{11}{36} \\ 0 & 0 & 0 & 0 & \frac{25}{36} & \frac{11}{36} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} = \frac{1}{36} \begin{bmatrix} 1 & 3 & 5 & 7 & 9 & 11 \\ 0 & 4 & 5 & 7 & 9 & 11 \\ 0 & 0 & 9 & 7 & 9 & 11 \\ 0 & 0 & 0 & 16 & 9 & 11 \\ 0 & 0 & 0 & 0 & 25 & 11 \\ 0 & 0 & 0 & 0 & 0 & 36 \end{bmatrix}$$

Initial state probability distribution is $p^{(0)} = \left(\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}\right)$

Since all the values $1, 2, \ldots, 6$ are equally likely.

$$P(X_2=6) = P(X_2=6, X_0=1) + P(X_2=6, X_0=2) + P(X_2=6, X_0=3) + P(X_2=6, X_0=4) + P(X_2=6, X_0=5) + P(X_2=6, X_0=6)$$

$$= P(X_2=6/X_0=1) . P(X_0=1) + P(X_2=6/X_0=2) . P(X_0=2) + P(X_2=6(X_0=3) . P(X_0=3) + P(X_2=6/X_0=4) \cdot P(X_0=4) + P(X_2=6/X_0=5) \cdot P(X_0=5) + P(X_2=6/X_0=6) . P(X_0=6)$$

$$= P_{16}^{(2)} \cdot \frac{1}{6} + P_{26}^{(2)} \cdot \frac{1}{6} + P_{36}^{(2)} \cdot \frac{1}{6} + P_{46}^{(2)} \cdot \frac{1}{6} + P_{56}^{(2)} \cdot \frac{1}{6} + P_{66}^{(2)} \cdot \frac{1}{6}$$

$$= \frac{1}{6}\left[\frac{11}{36} + \frac{11}{36} + \frac{11}{36} + \frac{11}{36} + \frac{11}{36} + \frac{36}{36}\right]$$

$$= \frac{1}{6} \times \frac{1}{36}[91] = \frac{91}{216}$$

---

⑩ Find $P(n)$ for the homogeneous Markov chain with the following tpm

$$P = \begin{array}{c|cc} & 0 & 1 \\ \hline 0 & 1-a & a \\ 1 & b & 1-b \end{array}$$ where $0<a<1$, $0<b<1$.

Sol: Given $P = \begin{pmatrix} 1-a & a \\ b & 1-b \end{pmatrix}$

$p^{(n)} = p^{(n-1)} p$

$$P(n) = \begin{bmatrix} P_{00}(n-1) & P_{01}(n-1) \\ P_{10}(n-1) & P_{11}(n-1) \end{bmatrix} \begin{bmatrix} 1-a & a \\ b & 1-b \end{bmatrix}$$

Using Chapman-Kolmogorov thm.,

$$P_{00}(n) = (1-a)P_{00}(n-1) + b\,P_{01}(n-1) \quad , n>1$$

Since for all n, P(n-1) is a stochastic matrix, we have

$$P_{00}(n-1)+P_{01}(n-1)=1 \Rightarrow P_{01}(n-1)=1-P_{00}(n-1)$$

$$\therefore P_{00}(n)=(1-a)P_{00}(n-1)+b(1-P_{00}(n-1))$$

$$=(1-a)P_{00}(n-1)+b-bP_{00}(n-1)$$

$$=(1-a-b)P_{00}(n-1)+b \quad —①$$

Subst. n by n-1 in ①,

$$P_{00}(n-1)=(1-a-b)P_{00}(n-2)+b$$

∴ ① becomes, $P_{00}(n)=$ ~~P~~ $(1-a-b)\left[(1-a-b)P_{00}(n-2)+b\right]+b$

$$=(1-a-b)^2P_{00}(n-2)+b(1-a-b)+b$$

$$=b+b(1-a-b)+(1-a-b)^2P_{00}(n-2)$$

$$=b+b(1-a-b)+b(1-a-b)^2+\dots+(1-a-b)^{n-2}P_{00}(n-(n-2))$$

$$=b+b(1-a-b)+b(1-a-b)^2+\dots+(1-a-b)^{n-2}\left[(1-a-b)P_{00}(1)+b\right]$$

$$=b+b(1-a-b)+b(1-a-b)^2+\dots+b(1-a-b)^{n-2}+(1-a-b)^{n-1}(1-a)$$

$$=b\left[1+(1-a-b)+(1-a-b)^2+\dots+(1-a-b)^{n-2}\right]+(1-a)(1-a-b)^{n-1}$$

$$=b\left[\frac{1-(1-a-b)^{n-1}}{1-(1-a-b)}\right]+(1-a)(1-a-b)^{n-1} \quad \left[\begin{array}{c}\text{Sum of G.P (n terms)}\\ S_n=\frac{a(1-r^n)}{1-r}\end{array}\right]$$

$$=b\left[\frac{1-(1-a-b)^{n-1}}{a+b}\right]+(1-a)(1-a-b)^{n-1}$$

$$=\frac{b}{a+b}-\frac{b(1-a-b)^{n-1}}{a+b}+(1-a)(1-a-b)^{n-1}$$

$$=\frac{b}{a+b}+\frac{(-b)(1-a-b)^{n-1}+(a+b)(1-a)(1-a-b)^{n-1}}{a+b}$$

$$=\frac{b}{a+b}+\frac{(-b)(1-a-b)^{n-1}+(1-a-b)^{n-1}(a-a^2+b-ab)}{a+b}$$

$$=\frac{b}{a+b}+\frac{a(1-a-b)(1-a-b)^{n-1}}{a+b}$$

$$=\frac{b}{a+b}+\frac{a(1-a-b)^n}{a+b}$$

Since the required matrix is a stochastic matrix, we have

$$P_{00}(n)+P_{01}(n)=1 \Rightarrow P_{01}(n)=1-P_{00}(n)$$

$$=1-\left(\frac{b}{a+b}+\frac{a(1-a-b)^n}{a+b}\right)$$

$$=\frac{a+b-b-a(1-a-b)^n}{a+b}=\frac{a-a(1-a-b)^n}{a+b}$$

Similarly, $P_{10}(n) = (1-a)P_{10}(n-1) + b\,P_{11}(n-1)$

$$= (1-a)P_{10}(n-1) + b\left[1-P_{10}(n-1)\right] \quad (\because P_{10}(n-1)+P_{11}(n-1)=1)$$

$$= (1-a)P_{10}(n-1) + b - bP_{10}(n-1)$$

$$= (1-a-b)P_{10}(n-1)+b$$

$$= b+b(1-a-b)+b(1-a-b)^2+\cdots+(1-a-b)^{n-2}P_{10}(2)$$

$$= b+b(1-a-b)+b(1-a-b)^2+\cdots+(1-a-b)^{n-2}\left(b+(1-a-b)P_{10}(1)\right)$$

$$= b+b(1-a-b)+b(1-a-b)^2+\cdots+b(1-a-b)^{n-2}+(1-a-b)^{n-1}b$$

$$= b\left[1+(1-a-b)+(1-a-b)^2+\cdots+(1-a-b)^{n-1}\right]$$

$$= b\left[\frac{1-(1-a-b)^n}{1-(1-a-b)}\right] = b\left[\frac{1-(1-a-b)^n}{a+b}\right]$$

$$= \frac{b-b(1-a-b)^n}{a+b}$$

Also $P_{10}(n)+P_{11}(n)=1 \Rightarrow P_{11}(n) = 1-P_{10}(n)$

$$= 1-\left(\frac{b-b(1-a-b)^n}{a+b}\right)$$

$$= \frac{a+b-b+b(1-a-b)^n}{a+b}$$

$$= \frac{a+b(1-a-b)^n}{a+b}$$

$\therefore$ The required tpm is $P(n) = \begin{pmatrix} \dfrac{b+a(1-a-b)^n}{a+b} & \dfrac{a-a(1-a-b)^n}{a+b} \\ \dfrac{b-b(1-a-b)^n}{a+b} & \dfrac{a+b(1-a-b)^n}{a+b} \end{pmatrix}$

---

⑪ There are 2 white balls in bag A & 3 red balls in bag B. At each step of the process, a ball is selected from each bag & the 2 balls selected are interchanged. Let the state a_i of the system be the no. of red ball in A after i change. What is the probability that there are 2 red balls in A after 3 steps? In the long run, what is the probability that there are 2 red balls in bag A?

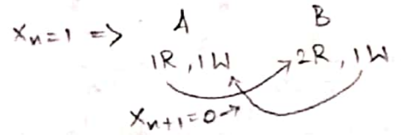
Sol: State space of the chain $\{X_n\} = (0,1,2)$ ($\because$ the no. of balls in the bag A is always 2)

The tpm of the chain $\{X_n\}$ be $P = \begin{array}{cc} & X_{n+1} \\ & \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ X_n \begin{array}{c} 0 \\ 1 \\ 2 \end{array} & \begin{bmatrix} P_{00} & P_{01} & P_{02} \\ P_{10} & P_{11} & P_{12} \\ P_{20} & P_{21} & P_{22} \end{bmatrix} \end{array}$

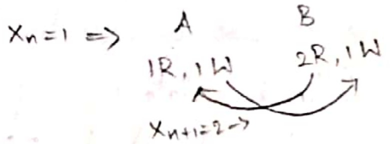
$P_{02} = P_{20} = 0$ $\quad$ $\left[\because P(\underset{\text{2 red balls}}{X_{n+1}=2} / \underset{\text{0 red ball}}{X_n=0}) = 0 \rightarrow\right.$ At the initial stage, there are 0 red ball in bag A, after the first transition bag A contains only one red ball. But here we find the prob. of 2 red balls in bag A, so the value of this prob. is zero]

Let $X_n = 1$

(ie) A contains 1 red ball (& 1 white ball) & B contains 1 white & 2 red balls.



$\therefore P[X_{n+1}=0 / X_n = 1] = P_{10} = \frac{1}{2} \times \frac{1}{3} = \frac{1}{6}$

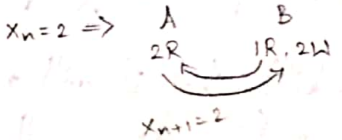


$P_{12} = P[X_{n+1} = 2 / X_n = 1] = \frac{1}{2} \times \frac{2}{3} = \frac{1}{3}$

Since P is a stochastic matrix,

$P_{10} + P_{11} + P_{12} = 1 \Rightarrow \frac{1}{6} + P_{11} + \frac{1}{3} = 1$

$\Rightarrow P_{11} = 1/2$



Similarly, $P_{20} + P_{21} + P_{22} = 1 \Rightarrow 0 + P_{21} + P_{22} = 1$

$P_{22} = P[X_{n+1} = 2 / X_n = 2] = \frac{2}{2} \times \frac{1}{3} = \frac{1}{3}$

$\therefore P_{21} + \frac{1}{3} = 1 \Rightarrow P_{21} = 2/3$ $\qquad$ $P_{00} = \{X_{n+1} = 0 / X_n = 0\} = 0$

$$\therefore P = \begin{pmatrix} 0 & 1 & 0 \\ 1/6 & 1/2 & 1/3 \\ 0 & 2/3 & 1/3 \end{pmatrix}$$

$P(X_0 = 0) = 1$, $P(X_0 = 1) = 0$, $P(X_0 = 2) = 0$

[Before interchanging, there are 0 red ball in bag A, so $P(X_0=0)=1$. There is no chance to get 1 or 2 red balls in bag A before interchanging]

(ie) $p^{(0)} = (1, 0, 0)$

$$p^{(1)} = p^{(0)}.P = (1\ 0\ 0)\begin{pmatrix} 0 & 1 & 0 \\ 1/6 & 1/2 & 1/3 \\ 0 & 2/3 & 1/3 \end{pmatrix} = (0\ 1\ 0)$$

$$p^{(2)} = p^{(1)}.P = (0\ 1\ 0)\begin{pmatrix} 0 & 1 & 0 \\ 1/6 & 1/2 & 1/3 \\ 0 & 2/3 & 1/3 \end{pmatrix} = \left(\frac{1}{6}\ \frac{1}{2}\ \frac{1}{3}\right)$$

$$p^{(3)} = p^{(2)}.P = \left(\frac{1}{6}\ \frac{1}{2}\ \frac{1}{3}\right)\begin{pmatrix} 0 & 1 & 0 \\ 1/6 & 1/2 & 1/3 \\ 0 & 2/3 & 1/3 \end{pmatrix} = \left(\frac{1}{12}\ \frac{23}{36}\ \frac{5}{18}\right)$$

$\therefore$ P(2 red balls in bag A after 3 steps) $= P(X_3 = 2) = \frac{5}{18}$.

Let the stationary probability distribution of the chain be $\pi = (\pi_0\ \pi_1\ \pi_2)$

By the property $\pi P = \pi$ & $\pi_0 + \pi_1 + \pi_2 = 1$

$$\text{(ie)}\ (\pi_0\ \pi_1\ \pi_2)\begin{pmatrix} 0 & 1 & 0 \\ 1/6 & 1/2 & 1/3 \\ 0 & 2/3 & 1/3 \end{pmatrix} = (\pi_0\ \pi_1\ \pi_2)$$

$\Rightarrow \frac{\pi_1}{6} = \pi_0$, $\pi_0 + \frac{\pi_1}{2} + \frac{2\pi_2}{3} = \pi_1$, $\frac{\pi_1}{3} + \frac{\pi_2}{3} = \pi_2$ & $\pi_0 + \pi_1 + \pi_2 = 1$

$\Rightarrow 6\pi_0 - \pi_1 = 0$, $6\pi_0 + 3\pi_1 + 4\pi_2 - 6\pi_1 = 0$, $\pi_1 + \pi_2 - 3\pi_2 = 0$, $\pi_0 + \pi_1 + \pi_2 = 1$

$\Rightarrow 6\pi_0 - \pi_1 = 0$ —①, $6\pi_0 - 3\pi_1 + 4\pi_2 = 0$ —②, $\pi_1 - 2\pi_2 = 0$ —③, $\pi_0 + \pi_1 + \pi_2 = 1$ —④

① $\Rightarrow 6\pi_0 = \pi_1 \Rightarrow \pi_0 = \pi_1/6$

③ $\Rightarrow \pi_1 = 2\pi_2 \Rightarrow \pi_2 = \pi_1/2$

$\therefore$ ④ becomes, $\frac{\pi_1}{6} + \pi_1 + \frac{\pi_1}{2} = 1 \Rightarrow \pi_1 + 6\pi_1 + 3\pi_1 = 6 \Rightarrow 10\pi_1 = 6 \Rightarrow \pi_1 = 3/5$

$\therefore \pi_0 = \frac{3}{5\times6} = \frac{3}{30} = \frac{1}{10}$; $\pi_2 = \frac{3}{5\times2} = \frac{3}{10}$

$\therefore$ In the long run, the prob/. to get 2 red balls in bag A is $3/10$.

⑫ A raining process is considered as two state Markov chain. If it rains, it is considered to be state 0 & if it does not rain, the chain is in state 1. The transition probability of the Markov chain is defined as $P = \begin{bmatrix} 0.6 & 0.4 \\ 0.2 & 0.8 \end{bmatrix}$. Find the probability that it will rain for 3 days from today assuming that it ~~will rain~~ is raining today. after ~~3 days~~. Assume the initial probabilities of state 0 & state 1 as 0.4 & 0.6 respectively. Find also the unconditional probability that it will rain after three days.

Sol: Given $P = \begin{bmatrix} 0.6 & 0.4 \\ 0.2 & 0.8 \end{bmatrix}$, $P(X_0=0)=0.4$, $P(X_0=1)=0.6$

$$P^2 = \begin{pmatrix} 0.6 & 0.4 \\ 0.2 & 0.8 \end{pmatrix}\begin{pmatrix} 0.6 & 0.4 \\ 0.2 & 0.8 \end{pmatrix} = \begin{pmatrix} 0.44 & 0.56 \\ 0.28 & 0.72 \end{pmatrix}$$

$$P^3 = \begin{pmatrix} 0.44 & 0.56 \\ 0.28 & 0.72 \end{pmatrix}\begin{pmatrix} 0.6 & 0.4 \\ 0.2 & 0.8 \end{pmatrix} = \begin{pmatrix} 0.376 & 0.624 \\ 0.312 & 0.688 \end{pmatrix}$$

The probability that it will rain on third day given that it will rain today is 0.376.

The unconditional probability that it will rain after three days is

$$P[X_3=0] = P[X_3=0, X_0=0] + P[X_3=0, X_0=1]$$

$$= P[X_3=0/X_0=0].P[X_0=0] + P[X_3=0/X_0=1].P[X_0=1]$$

$$= P_{00}^{(3)}.P(X_0=0) + P_{10}^{(3)}.P(X_0=1)$$

$$= (0.376)(0.4) + (0.312)(0.6) = 0.3376$$

⑬ A person owning a scooter has the option to switch over to scooter, bike or a car next time with the probability of $(0.3, 0.5, 0.2)$. If the tpm is $\begin{bmatrix} 0.4 & 0.3 & 0.3 \\ 0.2 & 0.5 & 0.3 \\ 0.25 & 0.25 & 0.5 \end{bmatrix}$. What are the probabilities vehicles related to his fourth purchase?

$p^{(1)} = p^{(0)}.P = (0.27 \ 0.39 \ 0.34)$

$p^{(2)} = p^{(1)}.P = (0.271 \ 0.361 \ 0.368)$

$p^{(3)} = p^{(2)}.P = (0.2726 \ 0.3538 \ 0.3736)$

$\therefore$ Prob/. of fourth purchase $= (0.2726 \ 0.3538 \ 0.3736)$

Sol: Given $P = \begin{bmatrix} 0.4 & 0.3 & 0.3 \\ 0.2 & 0.5 & 0.3 \\ 0.25 & 0.25 & 0.5 \end{bmatrix}$

$$P^2 = \begin{pmatrix} 0.4 & 0.3 & 0.3 \\ 0.2 & 0.5 & 0.3 \\ 0.25 & 0.25 & 0.5 \end{pmatrix}\begin{pmatrix} 0.4 & 0.3 & 0.3 \\ 0.2 & 0.5 & 0.3 \\ 0.25 & 0.25 & 0.5 \end{pmatrix} = \begin{pmatrix} 0.295 & 0.345 & 0.36 \\ 0.255 & 0.385 & 0.36 \\ 0.275 & 0.325 & 0.4 \end{pmatrix}$$

$$P^3 = \begin{pmatrix} 0.295 & 0.345 & 0.36 \\ 0.255 & 0.385 & 0.36 \\ 0.275 & 0.325 & 0.4 \end{pmatrix} \begin{pmatrix} 0.4 & 0.3 & 0.3 \\ 0.2 & 0.5 & 0.3 \\ 0.25 & 0.25 & 0.5 \end{pmatrix} = \begin{pmatrix} 0.277 & 0.351 & 0.372 \\ 0.269 & 0.359 & 0.372 \\ 0.275 & 0.345 & 0.38 \end{pmatrix}$$

Probabilities of his fourth purchases $= (0.3 \;\; 0.5 \;\; 0.2)\, P^3$

$$= (0.3 \;\; 0.5 \;\; 0.2) \begin{pmatrix} 0.277 & 0.351 & 0.372 \\ 0.269 & 0.359 & 0.372 \\ 0.275 & 0.345 & 0.38 \end{pmatrix}$$

$$= (0.2726 \;\; 0.3538 \;\; 0.3736)$$

⑭ An Engineer analyzing a series of digital signals generated by a testing system observes that only 1 out of 15 highly distorted signals followed a highly distorted signal with no recognizable signal, whereas 20 out of 23 recognizable signals follow recognizable signals with no highly distorted signals between. Given that only highly distorted signals are not recognizable. Find the fraction of signals that are highly distorted.

Sol: If $n \geq 1$, $X_n = 1$, if the n^{th} signal generated is highly distorted.

$X_n = 0$, if the n^{th} signal generated is recognizable.

Then, clearly $\{X_n ; n = 0, 1, \dots\}$ is a Markov chain with state space $\{0, 1\}$ & the tpm is given by

$$P = \begin{array}{c|c} & \\ X_n \; \begin{matrix} 0 \\ 1 \end{matrix} & \begin{pmatrix} \frac{20}{23} & \frac{3}{23} \\ \frac{14}{15} & \frac{1}{15} \end{pmatrix} \end{array}$$

$\pi_0 \to$ the fraction of signals that are recognizable.

$\pi_1 \to$ the fraction of signals that are highly distorted.

$$(\pi_0 \;\; \pi_1) \begin{pmatrix} \frac{20}{23} & \frac{3}{23} \\ \frac{14}{15} & \frac{1}{15} \end{pmatrix} = (\pi_0 \;\; \pi_1)$$

$$\Rightarrow \frac{20}{23}\pi_0 + \overset{14/15}{\cancel{\quad}}\pi_1 = \pi_0 \Rightarrow \pi_1 = \frac{15}{14}\left(\pi_0 - \frac{20}{23}\pi_0\right) = \frac{15}{14} \times \frac{3\pi_0}{23}$$

$$\frac{3}{23}\pi_0 + \frac{1}{15}\pi_1 = \pi_1 \quad \& \quad \pi_0 + \pi_1 = 1$$

$$\pi_0 + \pi_1 = 1 \Rightarrow \pi_0 + \frac{45\pi_0}{322} = 1 \Rightarrow \pi_0\left(1 + \frac{45}{322}\right) = 1 \Rightarrow \pi_0 = \frac{322}{367} = 0.8774$$

$$\pi_1 = 1 - \pi_0 = 1 - 0.8774 = 0.1226$$

∴ 12.26% of the signals generated by the testing system are highly distorted.

⑮ A man either drives a car or catches a train to go to office each day. It never goes 2 days in a row by train but if he drives one day, then the next day he is just as likely to drive again as he is to travel by train. Now suppose that on the first day of the week, the man tossed a fair die & drove to work iff a 6 appeared. Find (1) the probability that he takes a train on the 3rd day.
(2) the probability that he drives to work in the long run.

Sol: The travel pattern is a Markov chain, with state space = (Train, car)

The tpm of the chain is
$$P = \begin{array}{cc} & \begin{array}{cc} & X_{n+1} \\ T & C \end{array} \\ \begin{array}{c} T \\ X_n\ C \end{array} & \begin{pmatrix} 0 & 1 \\ 1/2 & 1/2 \end{pmatrix} \end{array}$$

The initial state probability distribution is $p^{(1)} = \left(\frac{5}{6}, \frac{1}{6}\right)$,

since P(travelling by car) = P(getting 6 in the toss of the die) $= \frac{1}{6}$

& P(travelling by train) $= \frac{5}{6}$

$$p^{(2)} = p^{(1)}P = \left(\frac{5}{6} \ \ \frac{1}{6}\right)\begin{pmatrix} 0 & 1 \\ 1/2 & 1/2 \end{pmatrix} = \left(\frac{1}{12} \ \ \frac{11}{12}\right)$$

$$p^{(3)} = p^{(2)}P = \left(\frac{1}{12} \ \ \frac{11}{12}\right)\begin{pmatrix} 0 & 1 \\ 1/2 & 1/2 \end{pmatrix} = \left(\frac{11}{24} \ \ \frac{13}{24}\right)$$

∴ P(the man travels by train on the third day) $= \frac{11}{24}$

Let $\pi = (\pi_1, \pi_2)$ be the limiting form of the state probability distribution of the Markov chain. By the property of π, $\pi P = \pi$

$$(ie)\ (\pi_1 \ \ \pi_2)\begin{pmatrix} 0 & 1 \\ 1/2 & 1/2 \end{pmatrix} = (\pi_1 \ \ \pi_2)$$

$$\Rightarrow \frac{\pi_2}{2} = \pi_1 \quad \& \quad \pi_1 + \frac{\pi_2}{2} = \pi_2 \ , \quad \pi_1 + \pi_2 = 1$$

$$\pi_1 + \pi_2 = 1 \Rightarrow \frac{\pi_2}{2} + \pi_2 = 1 \Rightarrow \pi_2\left(1 + \tfrac{1}{2}\right) = 1 \Rightarrow \pi_2\left(\tfrac{3}{2}\right) = 1 \Rightarrow \pi_2 = 2/3$$

$$\pi_1 + \pi_2 = 1 \Rightarrow \pi_1 = 1 - 2/3 = 1/3$$

∴ P(the man travels by car in the long run) $= \frac{2}{3}$

Poisson Process:

A random process $X(t, s)$ is called as discrete random process if the random variable X takes only the discrete values while t is continuous is called the Poisson process if it satisfy the following postulates.

(i) $P[1 \text{ occurrence exactly}] = \lambda h + o(h)$

(ii) $P[\text{zero occurrence}] = 1 - \lambda h + o(h)$

(iii) $P[\text{more than one occurrences}] = o(h)$

(iv) The times of event occurrences are satistically independent. So, the no. of occurrences over a time interval is independent of the no. of occurrences in any other non-overlapping time interval.

(v) The no. of occurrences over a period $(t_0, t_0+$

② The sum of two independent Poisson process is again a Poisson process. (Additive property).

Let $x(t) = x_1(t) + x_2(t)$

$$P[x(t)=n] = \sum_{r=0}^{n} P[x_1(t)=r] \cdot P[x_2(t)=n-r]$$

$$= \sum_{r=0}^{n} \frac{e^{-\lambda_1 t}(\lambda_1 t)^r}{r!} \frac{e^{-\lambda_2 t}(\lambda_2 t)^{n-r}}{(n-r)!}$$

$$= \frac{e^{-(\lambda_1+\lambda_2)t}}{n!} \sum_{r=0}^{n} \frac{n!}{r!(n-r)!} (\lambda_1 t)^r (\lambda_2 t)^{n-r}$$

$$= \frac{e^{-(\lambda_1+\lambda_2)t}}{n!} \sum_{r=0}^{n} nc_r (\lambda_1 t)^r (\lambda_2 t)^{n-r}$$

$$= \frac{e^{-(\lambda_1+\lambda_2)t}}{n!} [\lambda_1 t + \lambda_2 t]^n = \frac{e^{-(\lambda_1+\lambda_2)t}}{n!} ((\lambda_1+\lambda_2)t)^n$$

Hence $\{x_1(t)+x_2(t)\}$ is a Poisson process with parameter $(\lambda_1+\lambda_2)t$.

③ The difference of two independent Poisson processes is not a Poisson process.

Let $x(t) = x_1(t) - x_2(t)$

Then $E[x(t)] = E[x_1(t) - x_2(t)] = E[x_1(t)] - E[x_2(t)]$

$$= \lambda_1 t - \lambda_2 t = (\lambda_1 - \lambda_2)t$$

$$E[x^2(t)] = E[(x_1(t) - x_2(t))^2]$$

$$= E[x_1^2(t)] + E[x_2^2(t)] - 2E[x_1(t)\, x_2(t)]$$

$$= E[x_1^2(t)] + E[x_2^2(t)] - 2E[x_1(t)]E[x_2(t)] \quad (\because x_1(t) \,\&\, x_2(t) \text{ are independent})$$

$$= \lambda_1^2 t^2 + \lambda t + \lambda_2^2 t^2 + \lambda_2 t - 2\lambda_1 t \cdot \lambda_2 t$$

$$= \lambda_1^2 t^2 + \lambda_1 t + \lambda_2^2 t^2 + \lambda_2 t - 2\lambda_1\lambda_2 t^2$$

$$= t^2(\lambda_1^2 + \lambda_2^2 - 2\lambda_1\lambda_2) + (\lambda_1+\lambda_2)t$$

$$= (\lambda_1-\lambda_2)^2 t^2 + (\lambda_1+\lambda_2)t$$

$$\neq (\lambda_1-\lambda_2)^2 t^2 + (\lambda_1-\lambda_2)t$$

$\therefore$ $x(t) = x_1(t) - x_2(t)$ is not a Poisson process.

④ The inter arrival time of a Poisson process with parameter λ has an exponential distribution with mean $\frac{1}{\lambda}$.

Consider two consecutive occurrences of the event E_i & E_{i+1}.

Let E_i takes place at time t_i. Let T the interval between the occurrences of E_i & E_{i+1}. Then T is a continuous rv.

E_i & E_{i+1} are 2 consecutive events
There is no other event between them
E_{i+1} occurs only in time t_i+T

$P(T>t) = P[E_{i+1}$ does not occur in $(t_i, t_i+t)]$

$= P[$No event occurs in an interval of length $t]$

$$= P[X(t)=0] = P_0(t) = \frac{e^{-\lambda t}(\lambda t)^0}{0!} = e^{-\lambda t}$$

The cumulative distribution of T is given by

$F(t) = P(T \le t) = 1 - P(T>t) = 1-e^{-\lambda t}$

$\therefore$ The probability density funct. of T is given by

$$f(t) = \frac{d}{dt}[F(t)] = \frac{d}{dt}[1-e^{-\lambda t}] = \lambda e^{-\lambda t} \quad (t \ge 0)$$

which is an exponential distribution with mean $1/\lambda$.

---

⑥ If the no. of occurrences of an event E in an interval of length t is a Poisson process $\{X(t)\}$ with parameter λt & if each occurrence of E has a constant probability p of being recorded & the recording are independent of each other, then the no. $N(t)$ of the recorded occurrences in t is also a Poisson process with parameter $\lambda p t$.

$P[N(t)=n] = \sum_{r=0}^{\infty} P[E$ occurs $(n+r)$ times in t & n of them are recorded$]$

$$= \sum_{r=0}^{\infty} \frac{e^{-\lambda t}(\lambda t)^{n+r}}{(n+r)!} (n+r)C_n p^n q^r, \quad q=1-p$$

$$= e^{-\lambda t}(\lambda p t)^n \sum_{r=0}^{\infty} \frac{(\lambda q t)^r}{(n+r)!} \frac{(n+r)!}{n!\,r!}$$

$$= \frac{e^{-\lambda t}(\lambda p t)^n}{n!} \sum_{r=0}^{\infty} \frac{(\lambda q t)^r}{r!}$$

$$= \frac{e^{-\lambda t}(\lambda p t)^n}{n!} e^{\lambda q t} = \frac{e^{-\lambda(1-q)t}(\lambda p t)^n}{n!}$$

$$= \frac{e^{-\lambda p t}(\lambda p t)^n}{n!}$$

$\therefore N(t)$ is a Poisson random process.

---

⑦ If $\{X(t)\}$ is a Poisson process, prove that $P[X(s)=r / X(t)=n] = nC_r \left(\frac{s}{t}\right)^r \left(1-\frac{s}{t}\right)^{n-r}$ where $x<1$.

Proof: $$P[X(s)=r / X(t)=n] = \frac{P[(X(s)=r) \cap (X(t)=n)]}{P[X(t)=n]}$$

$$= \frac{P[X(s)=r \cap X(t-s)=n-r]}{P[X(t)=n]}$$

$$= \frac{P[X(s)=r].P[X(t-s)=n-r]}{P[X(t)=n]} \quad \text{(by independent)}$$

$$= \frac{\left(\frac{e^{-\lambda s}(\lambda s)^r}{r!}\right)\left(\frac{e^{-\lambda(t-s)}(\lambda(t-s))^{n-r}}{(n-r)!}\right)}{\frac{e^{-\lambda t}(\lambda t)^n}{n!}}$$

$$= \frac{n!}{r!(n-r)!} \frac{s^r (t-s)^{n-r}}{t^n} = nC_r\left(\frac{s}{r}\right)^r\left(1-\frac{s}{t}\right)^{n-r}$$

Problems:

① If customers arrive at a counter in accordance with a Poisson process with a mean rate of 3 per minute, find the probability that the interval between 2 consecutive arrivals is (i) more than 1 min. (ii) between 1 minute & 2 minutes (iii) 4 minutes or less.

Sol: By the property of Poisson process, the interval T between 2 consecutive follows an exponential distribution with parameter $\lambda = 3$.

(i) $P(T>1) = 3\int_1^\infty e^{-3t}\,dt \quad (\because f(t) = 3e^{-3t})$

$$= 3\left(\frac{e^{-3t}}{-3}\right)_1^\infty = -(0-e^{-3}) = e^{-3}$$

(ii) $P(1<T<2) = \int_1^2 3e^{-3t}\,dt = 3\left(\frac{e^{-3t}}{-3}\right)_1^2 = -(e^{-6}-e^{-3}) = e^{-3}-e^{-6}$

(iii) $P(T\leq 4) = \int_0^4 3e^{-3t}\,dt = 3\left(\frac{e^{-3t}}{-3}\right)_0^4 = -(e^{-12}-1) = 1-e^{-12}$

② If $\{X(t)\}$ & $\{Y(t)\}$ are two independent Poisson processes, show that the conditional distribution of $\{X(t)\}$ given $\{X(t)+Y(t)\}$ is binomial.

Sol: Let $\{X(t)\}$ & $\{Y(t)\}$ be two independent Poisson processes with parameter $\lambda_1 t$ & $\lambda_2 t$ respectively. Hence $\{X(t)+Y(t)\}$ is also a Poisson process with parameter $\lambda_1 t + \lambda_2 t$. (By additive property)

$$P[X(t)=m / X(t)+Y(t)=n] = \frac{P[(X(t)=m)\cap(X(t)+Y(t)=n)]}{P[X(t)+Y(t)=n]}$$

$$= \frac{P[(X(t)=m)\cap(Y(t)=n-m)]}{P[X(t)+Y(t)=n]}$$

Since $\{X(t)\}$ & $\{Y(t)\}$ are independent, we get

$$P[X(t)=m \mid X(t)+Y(t)=n] = \frac{P[X(t)=m]\cdot P[Y(t)=n-m]}{P[X(t)+Y(t)=n]}$$ (∵ If A&B are independent, $P(A\cap B)=P(A)P(B)$)

$$= \frac{\dfrac{(\lambda_1 t)^m e^{-\lambda_1 t}}{m!} \cdot \dfrac{e^{-\lambda_2 t}(\lambda_2 t)^{n-m}}{(n-m)!}}{\dfrac{(\lambda_1+\lambda_2)^n t^n e^{-(\lambda_1+\lambda_2)t}}{n!}}$$

$$= \frac{n!}{m!(n-m)!} \frac{(\lambda_1)^m(\lambda_2)^{n-m}}{(\lambda_1+\lambda_2)^m(\lambda_1+\lambda_2)^{n-m}}$$ (∵ Sum of two Poisson process is again a Poisson process)

$$= \frac{n!}{m!(n-m)!}\left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^m\left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)^{n-m}$$

$= nC_m\, p^m q^{n-m}$ where $p = \frac{\lambda_1}{\lambda_1+\lambda_2}$ & $q = \frac{\lambda_2}{\lambda_1+\lambda_2}$. Note that $p+q=1$

Hence the conditional distribution of $\{X(t)\}$ given $\{X(t)+Y(t)\}$ is binomial.

---

③ A hard disk fails in a computer system & it follows a Poisson distribution with mean rate of 1 per week. Find the probability that 2 weeks have elapsed since last failure. If we have 5 extra hard disks & the next supply is not due in 10 weeks, find the probability that the machine will not be out of order in the next 10 weeks.

Sol: Mean failure rate $\lambda = 1$

$$P[\text{no failure in the 2 weeks}] = P[X(2)=0] = \frac{e^{-2\lambda}(2\lambda)^0}{0!} = e^{-2\lambda} = e^{-2} \quad (\because \lambda=1)$$

$$= 0.1353$$

Number of extra hard disks $= 5$

The ~~hard disk~~ machine will not be out of order in the next 10 weeks is given by

$$P[X(10)\le 5] = \sum_{n=0}^{5}\frac{e^{-10}10^n}{n!} = e^{-10} + \frac{e^{-10}.10}{1!} + \frac{e^{-10}10^2}{2!} + \frac{e^{-10}10^3}{3!} + \frac{e^{-10}10^4}{4!} + \frac{e^{-10}10^5}{5!}$$

$$= e^{-10}\left[1+10+\frac{100}{2}+\frac{1000}{6}+\frac{10000}{24}+\frac{100000}{120}\right]$$

$$= 0.0671$$

---

④ Queries presented in a Computer database are following a Poisson process of rate $\lambda = 6$ queries per minute. An experiment consists of monitoring the database for m minutes & recording N(m) the no. of queries presented.

(i) What is the probability that no queries in a one minute interval?

(ii) What is the probability that exactly 6 queries arriving in one minute interval?

(iii) What is the probability of less than 3 queries arriving in a half minute interval?

Sol: Let N(m) be the no. of queries presented in m minutes.

$$P[N(m)=x] = \frac{e^{-6m}(6m)^x}{x!}, \quad x=0,1,2,\ldots$$

$$(i)\ P[N(1)=0] = \frac{e^{-6(1)}(6(1))^0}{0!} = e^{-6} = 0.0025$$

$$(ii)\ P[N(1)=6] = \frac{e^{-6}(6)^6}{6!} = 0.1606$$

$$(iii)\ P[N(1/2)<3] = P[N(1/2)=0] + P[N(1/2)=1] + P[N(1/2)=2]$$

$$= \frac{e^{-3}(3)^0}{0!} + \frac{e^{-3}(3)^1}{1!} + \frac{e^{-3}(3)^2}{2!}$$

$$= e^{-3}\left[1+3+\frac{9}{2}\right] = 0.4232$$

⑤ VLSI chips, essential to the running of a computer system, fail in accordance with a Poisson distribution with the rate of one chip in about 5 weeks. If there are two spare chips on hand, & if a new supply will arrive in 8 weeks, what is the probability that during the next 8 weeks the system will be down for a week or more, owing to the lack of chips?

Sol: Let X(t) be the no. of failure of the VLSI chips measured in t weeks with rate $\lambda = 1/5 = 0.2$. Since X(t) is a Poisson process, WKT

$$P[X(t)=k] = \frac{e^{-0.2t}(0.2t)^k}{k!}, \quad k=0,1,2,\ldots$$

Since there are only two spare VLSI chips on hand, the probability that the system will be down for atleast one week before new supply in 8 weeks is given by $P[X(7)>2] = 1-P[X(7)\leq 2]$

$$= 1-\left[P[X(7)=0]+P[X(7)=1]+P[X(7)=2]\right]$$

$$= 1-\left[\frac{e^{-1.4}(1.4)^0}{0!} + \frac{e^{-1.4}(1.4)}{1!} + \frac{e^{-1.4}(1.4)^2}{2!}\right]$$

$$= 1-e^{-1.4}\left[1+1.4+0.98\right] = 0.1665$$

⑥ The no. of telephone calls arriving at a certain switch board within a time interval of length (measured in minutes) is a Poisson process $X(t)$ with parameter $\lambda=2$. Find the probability of (i) No telephone calls arriving at this switch board during a 5 minutes period. (ii) More than one telephone call arriving at this switch board during a given ½ minute period.

Sol: Since the Poisson process $X(t)$ has parameter $\lambda=2$, wkt

$$P[X(t)=k]=\frac{e^{-2t}(2t)^k}{k!}, k=0,1,2,\ldots$$

(i) $P[X(5)=0]=\frac{e^{-10}(10)^0}{0!}=e^{-10}=0.000045$

(ii) $P[X(0.5)\geq 1]=1-P[X(0.5)\leq 1]=1-(P[X(0.5)=0]+P[X(0.5)=1])$

$$=1-\left(\frac{e^{-1}(1)^0}{0!}+e^{-1}\right)=1-2e^{-1}$$

---

⑦ A fisherman catches fish at a Poisson rate of 2 per hour from a large pond with lots of fish. If he starts fishing at 10.00 a.m., What is the probability that he catches one fish by 10.30 a.m. & three fish by noon?

Sol: $X(t)$ is a Poisson process has the rate of $\lambda=2$ per hour. Thus we have

$$P[X(t)=k]=\frac{e^{-2t}(2t)^k}{k!}, k=0,1,2,\ldots$$

It is given that the fisherman starts fishing at 10.00 a.m. Thus the probability that he catches one fish by 10.30 a.m (in half-an-hour period) is given by $P[X(0.5)=1]=e^{-1}.1=e^{-1}=0.3679$

Similarly, the probability that the fisherman catches three fish by noon (in $\overset{2}{4}$ hours period) is given by

$$P[X(\overset{2}{4})=3]=\frac{e^{-8}8^3}{3!}=0.0286$$

A radar emits particles at the rate of 5 per minute according to Poisson distribution. Each particles emitted has probability 0.6. Find the probability that 10 particles are emitted in a 4 minute period.

Sol: Given $\lambda=5$ & $p=0.6$

By the property, the no. $N(t)$ of the recorded occurrences in t is a Poisson process with parameter λpt.

$$P[N(t)=n]=\frac{e^{-\lambda pt}(\lambda pt)^n}{n!}$$

$$P[N(4)=10]=\frac{e^{-12}(12)^{10}}{10!}$$

Probability law for the Poisson process $\{X(t)\}$:

Let λ be the rate of occurrences or no/. of occurrences per unit time & $P_m(t)$ be the probability of m occurrences of the event in the interval $(0,t)$. Hence $P_0(t)$ denotes the probability of having zero occurrence in the interval $(0,t)$. Now, the probability of zero occurrence of the event in the time interval 0 to $t+\Delta t$ is given by $P_0(t+\Delta t)$.

Also $P_0(t+\Delta t)$ can be written as

$P_0(t+\Delta t)$ = Probability of zero occurrence in the interval $(0,t)$ & also in the interval $(t, t+\Delta t)$.

(ie) $P_0(t+\Delta t) = P_0(t)[1-\lambda\Delta t]$ $(\because P\{0 \text{ occurrence in } (t, t+\Delta t)\} = 1-\lambda\Delta t)$

$1-\lambda h$

$$\Rightarrow P_0(t+\Delta t) = P_0(t) - \lambda P_0(t)\,\Delta t$$

$$\Rightarrow \frac{P_0(t+\Delta t) - P_0(t)}{\Delta t} = -\lambda P_0(t)$$

Taking limit as $\Delta t \to 0$, we get

$$\lim_{\Delta t \to 0} \frac{P_0(t+\Delta t) - P_0(t)}{\Delta t} = \lim_{\Delta t \to 0} -\lambda P_0(t) = -\lambda P_0(t) \text{ --- ①}$$

But WKT $\frac{dP_0}{dt} = \lim_{\Delta t \to 0} \frac{P_0(t+\Delta t) - P_0(t)}{\Delta t}$ --- Ⓐ

$$\therefore \frac{dP_0(t)}{dt} = -\lambda P_0(t) \quad (\text{using Ⓐ in ①})$$

$$\text{(ie)} \quad \frac{dP_0(t)}{P_0(t)} = -\lambda dt$$

Integrating we get, $\int \frac{dP_0(t)}{P_0(t)} = -\lambda \int dt$

$$\log P_0(t) = -\lambda t + c \quad \left(\text{using } \int \frac{dx}{x} = \log x + c\right)$$

$$\text{(ie)} \quad P_0(t) = e^{-\lambda t + c} = e^{-\lambda t} \cdot e^{c} = Ae^{-\lambda t}, \quad e^c = A \text{ --- ②}$$

Using $P_0(0) = 1$ in ②, we have

$$P_0(0) = A = 1$$

Substituting $A=1$ in ②, we get

$$\therefore P_0(t) = e^{-\lambda t} \text{ --- ③}$$

Now, $P_m(t) = P[x(t)=m]$ —④

$\therefore P_m(t+\Delta t) = P[x(t+\Delta t) = m]$

$= P[(m-1)$ occurrences in $(0,t)$ & 1 occurrences in $(t, t+\Delta t)]$

$+P[m$ occurrences in $(0,t)$ & no occurrences in $\underset{(1-\lambda\Delta t)}{\underset{\downarrow}{(t,t+\Delta t)}}]$

$$P_m(t+\Delta t) = P_{m-1}(t)(\lambda\Delta t) + P_m(t) - \lambda P_m(t)\Delta t$$

$$\therefore \frac{P_m(t+\Delta t) - P_m(t)}{\Delta t} = \lambda[P_{m-1}(t) - P_m(t)]$$

Taking the limits as $\Delta t \to 0$

$$\lim_{\Delta t\to 0} \frac{P_m(t+\Delta t) - P_m(t)}{\Delta t} = \lim_{\Delta t\to 0} \lambda[P_{m-1}(t) - P_m(t)]$$

$$\frac{dP_m(t)}{dt} = \lambda[P_{m-1}(t) - P_m(t)]$$

$$\frac{dP_m(t)}{dt} + \lambda P_m(t) = \lambda P_{m-1}(t) \quad —⑤$$

which is a linear differential equ. of first order in $P_m(t)$.

Here $P = \lambda$, $Q = \lambda P_{m-1}(t)$

Integrating factor of ⑤, if $\quad ye^{\int P dx} = \int Q e^{\int P dx} dx$

$$e^{\int P dt} = e^{\int \lambda dt} = e^{\lambda t}$$

$\therefore$ The solution of the equ. ⑤ is given by

$$P_m(t)e^{\lambda t} = \int_0^t \lambda P_{m-1}(t) e^{\lambda t} dt$$

$$\Rightarrow e^{\lambda t} P_m(t) = \lambda \int_0^t P_{m-1}(t) e^{\lambda t} dt$$

Taking $m=1$, we get

$$e^{\lambda t} P_1(t) = \lambda \int_0^t P_0(t) e^{\lambda t} dt \quad —⑥$$

From equ. ③, we have $P_0(t) = e^{-\lambda t}$

Subst. the value of $P_0(t)$ in ⑥, we get

$$e^{\lambda t} P_1(t) = \lambda \int_0^t e^{-\lambda t} e^{\lambda t} dt = \lambda [t]_0^t = \lambda t \Rightarrow P_1(t) = e^{-\lambda t} \lambda t \quad —⑦$$

Solving recursively, we get $P_2(t)$, $P_3(t)$, ... & so on.

In general, $P_m(t) = \frac{(\lambda t)^m e^{-\lambda t}}{m!}$, $m = 0, 1, 2, \ldots$

$\therefore$ P.d.f. of a Poisson process is

$$P_m(t) = \frac{(\lambda t)^m e^{-\lambda t}}{m!}, \quad m = 0, 1, 2, \ldots$$

(or)

$$P_x(t) = \frac{(\lambda t)^x e^{-\lambda t}}{x!}, \quad x = 0, 1, 2, \ldots$$

A radioactive source emits particles at a rate of 5 per minute in accordance with Poisson process. Each particles emitted has a probability 0.6 of being recorded. Find the probability that 10 particles are recorded in 4-min period.

Sol:

By the property of Poisson process, the no/. of recorded particles $N(t)$ is a Poisson process with parameter λp.

Here $\lambda = 5$, $p = 0.6$, $t = 4$

$$P(N(t) = r) = \frac{e^{-\lambda p t} (\lambda p t)^r}{r!}$$

$$P(N(4) = 10) = \frac{e^{-5(0.6)4} (5(0.6)4)^{10}}{10!} = \frac{e^{-12} (12)^{10}}{10!}$$

QUEUEING MODELS

Characteristics of Infinite capacity, single server poisson queue model I $(M/M/1):(\infty/FIFO)$, when $\lambda_n=\lambda$ & $\mu_n=\mu$ $(\lambda<\mu)$.

① Average no/. of customers in the system (L_s):

N - denotes the no/. of customers in the queueing system (ie, those in the queue & one who is being served.)

N is a discrete rv which can take values $0,1,2,\ldots\infty$

$$P(N=n)=P_n=\left(\frac{\lambda}{\mu}\right)^n\cdot P_o$$

We have $P_o=\dfrac{1}{1+\sum\limits_{n=1}^{\infty}\left(\frac{\lambda}{\mu}\right)^n}=\dfrac{1}{1+\frac{\lambda}{\mu}+\left(\frac{\lambda}{\mu}\right)^2+\ldots}=\dfrac{1}{\left(1-\frac{\lambda}{\mu}\right)^{-1}}=1-\dfrac{\lambda}{\mu}$

$$P_o=\frac{1}{\sum\limits_{n=0}^{\infty}\left(\frac{\lambda}{\mu}\right)^n}$$

$$\therefore P_n=\left(\frac{\lambda}{\mu}\right)^n\left(1-\frac{\lambda}{\mu}\right)$$

Now, $L_s=E(N)=\sum\limits_{n=0}^{\infty}nP_n$

$$=\sum_{n=0}^{\infty}n\left(\frac{\lambda}{\mu}\right)^n\left(1-\frac{\lambda}{\mu}\right)$$

$$=\left(1-\frac{\lambda}{\mu}\right)\left[\frac{\lambda}{\mu}+2\left(\frac{\lambda}{\mu}\right)^2+3\left(\frac{\lambda}{\mu}\right)^3+\ldots\right]$$

$$=\left(1-\frac{\lambda}{\mu}\right)\frac{\lambda}{\mu}\left[1+2\frac{\lambda}{\mu}+3\left(\frac{\lambda}{\mu}\right)^2+\ldots\right]$$

$$=\left(1-\frac{\lambda}{\mu}\right)\frac{\lambda}{\mu}\left[1-\frac{\lambda}{\mu}\right]^{-2}\quad\left(\because(1-x)^{-2}=1+2x+3x^2+\ldots\right)$$

$$=\frac{\lambda}{\mu}\left(1-\frac{\lambda}{\mu}\right)^{-1}=\frac{\lambda}{\mu}\left(\frac{\mu-\lambda}{\mu}\right)^{-1}$$

$$=\frac{\lambda}{\mu}\cdot\frac{\mu}{\mu-\lambda}=\frac{\lambda}{\mu-\lambda}$$

$$\therefore L_s=\frac{\lambda}{\mu-\lambda}$$

---

② The average no/. of customers in the queue (L_q):

If 'N' is the no/. of customers in the system, then the no/. of customers in the queue is $(N-1)$.

$$\therefore L_q = E(N-1) = \sum_{n=1}^{\infty} (n-1) P_n$$

$$= \sum_{n=1}^{\infty} (n-1)\left(\frac{\lambda}{\mu}\right)^n\left(1-\frac{\lambda}{\mu}\right)$$

$$=\left(1-\frac{\lambda}{\mu}\right)\sum_{n=1}^{\infty}(n-1)\left(\frac{\lambda}{\mu}\right)^n$$

$$=\left(1-\frac{\lambda}{\mu}\right)\left[\left(\frac{\lambda}{\mu}\right)^2+2\left(\frac{\lambda}{\mu}\right)^3+3\left(\frac{\lambda}{\mu}\right)^4+\cdots\right]$$

$$=\left(1-\frac{\lambda}{\mu}\right)\left(\frac{\lambda}{\mu}\right)^2\left[1+2\left(\frac{\lambda}{\mu}\right)+3\left(\frac{\lambda}{\mu}\right)^2+\cdots\right]$$

$$=\left(1-\frac{\lambda}{\mu}\right)\left(\frac{\lambda}{\mu}\right)^2\left[1-\frac{\lambda}{\mu}\right]^{-2}$$

$$=\left(1-\frac{\lambda}{\mu}\right)^{-1}\left(\frac{\lambda}{\mu}\right)^2=\left(\frac{\mu-\lambda}{\mu}\right)^{-1}\left(\frac{\lambda}{\mu}\right)^2$$

$$=\frac{\lambda^2}{\mu^2}\cdot\frac{\mu}{\mu-\lambda}=\frac{\lambda^2}{\mu(\mu-\lambda)}$$

③ Average no/. of customers in non-empty queues (L_w):

$L_w = E\{(N-1)|(N-1)>0\}$ since the queue is non-empty.

$$= \frac{E(N-1)}{P(N-1>0)} = \frac{E(N-1)}{P(N>1)} = \frac{\lambda^2}{\mu(\mu-\lambda)}\times\frac{1}{\sum_{n=2}^{\infty} P_n}$$

$$=\frac{\lambda^2}{\mu(\mu-\lambda)}\times\frac{1}{\sum_{n=2}^{\infty}\left(\frac{\lambda}{\mu}\right)^n\left(1-\frac{\lambda}{\mu}\right)}=\frac{\lambda^2}{\mu(\mu-\lambda)}\,\frac{1}{\left(1-\frac{\lambda}{\mu}\right)\sum_{n=2}^{\infty}\left(\frac{\lambda}{\mu}\right)^n}$$

$$=\frac{\lambda^2}{\mu(\mu-\lambda)}\,\frac{1}{\left(1-\frac{\lambda}{\mu}\right)\left[\left(\frac{\lambda}{\mu}\right)^2+\left(\frac{\lambda}{\mu}\right)^3+\left(\frac{\lambda}{\mu}\right)^4+\cdots\right]}$$

$$=\frac{\lambda^2}{\mu(\mu-\lambda)}\,\frac{1}{\left(\frac{\mu-\lambda}{\mu}\right)\left(\frac{\lambda}{\mu}\right)^2\left[1+\frac{\lambda}{\mu}+\left(\frac{\lambda}{\mu}\right)^2+\cdots\right]}$$

$$=\frac{\lambda^2}{\mu(\mu-\lambda)}\,\frac{\mu}{\mu-\lambda}\,\frac{\mu^2}{\lambda^2}\,\frac{1}{\left[1-\frac{\lambda}{\mu}\right]^{-1}}$$

$$=\frac{\mu^2}{(\mu-\lambda)^2}\left(1-\frac{\lambda}{\mu}\right)=\frac{\mu^2}{(\mu-\lambda)^2}\left(\frac{\mu-\lambda}{\mu}\right)=\frac{\mu}{\mu-\lambda}$$

④ The probability that the no/. of customers in the system exceeds k:

$$P(N>K) = \sum_{n=k+1}^{\infty} P_n = \sum_{n=K+1}^{\infty} \left(\frac{\lambda}{\mu}\right)^n \left(1-\frac{\lambda}{\mu}\right)$$

$$= \left(1-\frac{\lambda}{\mu}\right) \sum_{n=k+1}^{\infty} \left(\frac{\lambda}{\mu}\right)^n$$

$$= \left(1-\frac{\lambda}{\mu}\right)\left[\left(\frac{\lambda}{\mu}\right)^{K+1} + \left(\frac{\lambda}{\mu}\right)^{K+2} + \left(\frac{\lambda}{\mu}\right)^{K+3} + \cdots\right]$$

$$= \left(1-\frac{\lambda}{\mu}\right)\left(\frac{\lambda}{\mu}\right)^{K+1}\left[1+\frac{\lambda}{\mu}+\left(\frac{\lambda}{\mu}\right)^2+\cdots\right]$$

$$= \left(1-\frac{\lambda}{\mu}\right)\left(\frac{\lambda}{\mu}\right)^{K+1}\left[1-\frac{\lambda}{\mu}\right]^{-1}$$

$$= \left(\frac{\lambda}{\mu}\right)^{K+1}$$

---

⑤ The probability density function of the waiting time in the system:

W_s be continuous RV represents the waiting time of a customer in the system. Its Pdf be $f(\omega)$ & let $f(\omega/n)$ be density funf. of W_s, to the condition that n customers in the system when the customer arrives.

$$f(\omega) = \sum_{n=0}^{\infty} f(\omega/n) P_n$$

$f(\omega/n)$ = Pdf of sum of $(n+1)$ independent RVs with parameter μ.

$$= \frac{\mu^{n+1}}{n!} e^{-\mu\omega} \omega^n, \ \omega > 0$$

$$\therefore f(\omega) = \sum_{n=0}^{\infty} \frac{\mu^{n+1}}{n!} e^{-\mu\omega} \omega^n \left(\frac{\lambda}{\mu}\right)^n \left(1-\frac{\lambda}{\mu}\right)$$

$$= \sum_{n=0}^{\infty} \frac{\mu^{n+1}}{n!} e^{-\mu\omega} \omega^n \frac{\lambda^n}{\mu^n}\left(\frac{\mu-\lambda}{\mu}\right)$$

$$= e^{-\mu\omega}(\mu-\lambda)\sum_{n=0}^{\infty}\frac{\omega^n\lambda^n}{n!} = e^{-\mu\omega}(\mu-\lambda)\sum_{n=0}^{\infty}\frac{(\omega\lambda)^n}{n!}$$

$$= e^{-\mu\omega}(\mu-\lambda)\left[1+\frac{\omega\lambda}{1!}+\frac{(\omega\lambda)^2}{2!}+\cdots\right]$$

$$= e^{-\mu\omega}(\mu-\lambda)e^{\omega\lambda} \quad \left(\because 1+\frac{x}{1!}+\frac{x^2}{2!}+\cdots = e^x\right)$$

$$= (\mu-\lambda)e^{-\omega(\mu-\lambda)}$$ which is a exponential distribution with parameter $\mu-\lambda$.

---

⑥ The average waiting time of a customer in the system:

W_s follows exponential distribution with parameter $\mu-\lambda$.

$\therefore E(W_s) = \frac{1}{\mu-\lambda}$ ($\because$ The mean of an exponential distribution is the reciprocal of its parameter.)

⑦ The probability that the waiting time of a customer in the system exceeds t:

$$P(W_s>t) = \int_t^{\infty} f(\omega)\,d\omega$$

$$= \int_t^{\infty} (\mu-\lambda)e^{-(\mu-\lambda)\omega}\,d\omega = (\mu-\lambda)\left[\frac{e^{-(\mu-\lambda)\omega}}{-(\mu-\lambda)}\right]_t^{\infty}$$

$$= -\left[e^{-\infty} - e^{-(\mu-\lambda)t}\right] = e^{-(\mu-\lambda)t}$$

⑧ The probability density funf. of the waiting time (W_q) in the queue:

W_q represents the time between arrival & reach of service point.

Let Pdf of W_q be $g(\omega)$ & let $g(\omega/n)$ be the density funf. of W_q to the condition that there are 'n' customers in the system.

$g(\omega/n)$ = pdf of sum of n service times.

$$\therefore g(\omega/n) = \frac{\mu^n}{(n-1)!} e^{-\mu\omega} \omega^{n-1}, \quad \omega>0$$

$$\therefore g(\omega) = \sum_{n=1}^{\infty} \frac{\mu^n}{(n-1)!} e^{-\mu\omega} \omega^{n-1} \left(\frac{\lambda}{\mu}\right)^n \left(1-\frac{\lambda}{\mu}\right)$$

$$= e^{-\mu\omega}\left(1-\frac{\lambda}{\mu}\right) \sum_{n=1}^{\infty} \frac{\lambda^n \omega^{n-1}}{(n-1)!} = e^{-\mu\omega}\left(1-\frac{\lambda}{\mu}\right) \sum_{n=1}^{\infty} \frac{\lambda^{n-1+1}\omega^{n-1}}{(n-1)!}$$

$$= e^{-\mu\omega}\left(1-\frac{\lambda}{\mu}\right)\lambda \sum_{n=1}^{\infty} \frac{(\lambda\omega)^{n-1}}{(n-1)!}$$

$$= e^{-\mu\omega}\left(1-\frac{\lambda}{\mu}\right)\lambda \left[1+\frac{\lambda\omega}{1!}+\frac{(\lambda\omega)^2}{2!}+\cdots\right]$$

$$= e^{-\mu\omega}\left(1-\frac{\lambda}{\mu}\right)\lambda e^{\lambda\omega} = \lambda\left(1-\frac{\lambda}{\mu}\right)e^{-\omega(\mu-\lambda)}, \quad \omega>0$$

At $\omega=0$, $g(\omega) = \lambda\left(1-\frac{\lambda}{\mu}\right)$

⑨ The average waiting time of a customer in the queue:

$$E(W_q) = \int_0^{\infty} \omega f(\omega)\,d\omega = \int_0^{\infty} \omega\lambda\left(1-\frac{\lambda}{\mu}\right)e^{-\omega(\mu-\lambda)}\,d\omega$$

$$= \lambda\left(1-\frac{\lambda}{\mu}\right)\int_0^\infty w\, e^{-w(\mu-\lambda)}\, dw$$

$$= \lambda\left(1-\frac{\lambda}{\mu}\right)\left[w\,\frac{e^{-w(\mu-\lambda)}}{-(\mu-\lambda)} - \frac{e^{-w(\mu-\lambda)}}{(\mu-\lambda)^2}\right]_0^\infty$$

$$= \lambda\left(1-\frac{\lambda}{\mu}\right)\frac{1}{(\mu-\lambda)^2} = \lambda\left(\frac{\mu-\lambda}{\mu}\right)\frac{1}{(\mu-\lambda)^2} = \frac{\lambda}{\mu(\mu-\lambda)}$$

⑩ The average waiting time of a customer in the queue, if he has to wait

$$E[W_q / W_q > 0] = \frac{E(W_q)}{P(W_q > 0)} = \frac{E(W_q)}{1-P(W_q \leq 0)}$$

$$= \frac{E(W_q)}{1-P(W_q = 0)}$$

$$= \frac{E(W_q)}{1-P(\text{no customer in the queue})}$$

$$= \frac{E(W_q)}{1-P_0} = \frac{\lambda}{\mu(\mu-\lambda)} \cdot \frac{1}{1-\left(1-\frac{\lambda}{\mu}\right)}$$

$$= \frac{\lambda}{\mu(\mu-\lambda)} \cdot \frac{1}{\lambda/\mu} = \frac{\mu}{\mu(\mu-\lambda)} = \frac{1}{\mu-\lambda}$$

Little's formula:

① $E(N_s) = \frac{\lambda}{\mu-\lambda} = \lambda E(W_s)$ $(\because E(N_s) = L_s)$

② $E(N_q) = \frac{\lambda^2}{\mu(\mu-\lambda)} = \lambda E(W_q)$ $(\because E(N_q) = L_q)$

③ $E(W_s) = E(W_q) + \frac{1}{\mu}$

④ $E(N_s) = E(N_q) + \frac{\lambda}{\mu}$

Formulae:

① $\rho = \frac{\lambda}{\mu}$

② $P_0 = 1-\rho$

③ $P_n = \left(\frac{\lambda}{\mu}\right)^n\left(1-\frac{\lambda}{\mu}\right) = \rho^n P_0$ $(n = 0, 1, 2, \cdots, \infty)$

④ Little's formula:

$L_s = \frac{\lambda}{\mu-\lambda} = \frac{\rho}{1-\rho}$; $L_q = \rho L_s = L_s - \rho$; $W_s = \frac{1}{\lambda}L_s$; $W_q = \frac{1}{\lambda}L_q$

(OR)

$W_s = \frac{1}{\mu-\lambda}$; $W_q = W_s - \frac{1}{\mu}$; $L_s = \lambda W_s$; $L_q = \lambda W_q$

⑤ L_w (or) $L_n = \frac{\mu}{\mu-\lambda} = \frac{1}{\rho} L_s$

⑥ $W_n = \frac{1}{\mu-\lambda}$

⑦ $P(N \geq n) = \rho^n$

⑧ $P(N > k) = \rho^{k+1}$

⑨ $P(\omega) = (\mu-\lambda)e^{-(\mu-\lambda)\omega}$

⑩ $P(W_s > t) = e^{-(\mu-\lambda)t}$

Problems:

① Customers arrive at one-man barber shop according to a Poisson process with a mean interarrival time of 12 min. customers spend an average of 10 min. in the barber's chair.

(i) What is the expected no. of customers in the barber shop & in the queue? L_s & L_q

(ii) Calculate the % of time of arrival, can walk straight into the barber's chair without having to wait. P_0

(iii) How much time can customer expect to spend in the barber's shop? W_s

(iv) Management will provide another chair & here another barber, when a customer's waiting time in the shop exceeds 1.25h. How much must the average rate of arrivals increase to warrant a second barber?

(v) What is the average time customers spend in the queue? W_q

(vi) What is the prob. that the waiting time in the system is greater than 30 min.?

(vii) Calculate the % of customers who have to wait prior to getting into the barber's chair. $P(W>0)$

(viii) What is the prob. that more than 3 customers are in the system? $P(N>3)$

Sol: Given One man barber shop → Single server

Customers → Infinite capacity

∴ The given problem is (M/M/1) : (∞/FIFO) model.

Mean arrival rate $= \frac{1}{12}$ per min. $= \lambda$

Mean service rate $(\mu) = \frac{1}{10}$ per min.

$$\rho = \frac{\lambda}{\mu} = \frac{1/12}{1/10} = \frac{10}{12} = 5/6$$

(i) The expected no/. of customers in the barber shop:

$$L_s = \frac{\rho}{1-\rho} = \frac{5/6}{1-5/6} = \frac{5/6}{1/6} = 5$$

The expected no/. of customers in the queue:

$$L_q = \rho L_s = \frac{5}{6} \times 5 = \frac{25}{6} = 4.1667$$

(ii) P[a customer straight goes to the barber's chair] = P[No customer in the system]

$$= P_0 = 1-\rho = 1-\frac{5}{6} = 1/6 = 0.1667$$

∴ The percentage of time an arrival need not wait $= 0.1667 \times 100 = 16.67$

(iii) Expected time a customer spends in the barber's shop $= W_s$

$$= \frac{1}{\lambda} L_s = \frac{1}{1/12} \times 5 = 12 \times 5 = 60 \text{ mins.}$$

(iv) Given $W_s > 75$ mins

$$\frac{1}{\mu-\lambda} > 75 \Rightarrow \mu-\lambda < \frac{1}{75}$$

$$\Rightarrow \lambda > \mu - \frac{1}{75}$$

$$\Rightarrow \lambda > \frac{1}{10} - \frac{1}{75} \Rightarrow \lambda > \frac{13}{150}$$

Hence to warrant a second barber, the average arrival rate must increase by $\frac{13}{150} - \frac{1}{12} = \frac{1}{300}$ per min.

(v) Average waiting time per customer in the queue:

$$W_q = \frac{1}{\lambda} L_q = \frac{1}{1/12}\left(\frac{25}{6}\right) = 50 \text{ mins.}$$

(vi) The probability that the waiting in the system is greater than 30 mins.

$$= P[W_s > 30]$$

WKT $P[W_s > t] = e^{-(\mu-\lambda)t}$

$$P(W_s > 30) = e^{-\left(\frac{1}{10}-\frac{1}{12}\right)30} = e^{-0.5} = 0.6065$$

(vii) P[a customer has to wait] $= P[W > 0]$

$$= 1 - P[W \leq 0] = 1 - P[W = 0]$$

$$= 1 - P_0 = 1-(1-\rho) = \rho = \frac{5}{6}$$

∴ The percentage of customers who have to wait $= \frac{5}{6} \times 100 = 83.33$

(viii) Probability that more than k customers $= P(N>k) = \rho^{k+1}$

Probability that more than 3 customers $= P(N>3) = \rho^{3+1} = \rho^4 = \left(\frac{5}{6}\right)^4$

$$= 0.4823$$

⑧ If people arrive to purchase cinema tickets at the average rate of 6 per min., it takes an average of 7.5 seconds to purchase a ticket. If a person arrives 2 mins before the picture starts & if it takes exactly 1.5 min to reach the correct seat after purchasing the ticket.

(i) Can he expect to be seated for the start of the picture?

(ii) What is the probability that he will be seated for the start of the picture?

(iii) How early must he arrive in order to be 99% of sure of being seated for the start of the picture?

Sol: Given Ticket counter → Single server People → Infinite capacity

∴ The given problem is $(M/M/1):(\infty/FIFO)$ model.

Given Mean arrival rate $\lambda = 6$ per min.

Mean service rate $\mu = \frac{1}{7.5}$ per sec. $= \frac{1}{7.5} \times 60$ per min. $= 8$ per min.

$$\rho = \frac{\lambda}{\mu} = \frac{6}{8} = \frac{3}{4}$$

(i) Expected time a customer spends in the system $= W_s = \frac{1}{\mu - \lambda} = \frac{1}{8-6} = \frac{1}{2}$ min.

∴ E[The total time required to purchase the ticket & to reach the seat]

$$= \frac{1}{2} + 1\frac{1}{2} = 0.5 + 1.5 = 2 \text{ min}$$

Hence he can just be seated for the start of the picture.

(ii) P[he will be seated for the start of the picture]

$$= P[\text{total time} \leq 2 \text{ mins.}] = P[\overset{\rightarrow \text{waiting time}}{W} \leq 1/2]$$

$$= 1 - P[W > 1/2] = 1 - e^{-(8-6)\frac{1}{2}} \quad [\because P(W>t) = e^{-(\mu-\lambda)t}]$$

$$= 1 - e^{-1} = 0.6321$$

(iii) Given: $P[W \leq t] = 0.99$

$$1 - P[W>t] = 0.99 \Rightarrow P[W>t] = 1 - 0.99 = 0.01$$

$$\Rightarrow e^{-(\mu-\lambda)t} = 0.01$$

W_s = waiting time in the system = waiting time in the queue + ticket purchasing time

Take $\log_e$ on both sides

$$-(\mu-\lambda)t = \log_e 0.01$$

$$-(8-6)t = -4.6052$$

$$-2t = -4.6052 \Rightarrow t = \frac{4.6052}{2} = 2.3026 \text{ mins.}$$

∴ P[Ticket purchasing time ≤ 2.3] = 0.99

∴ P[Total time to get the ticket & to go to the seat ≤ (2.3+1.5)] = 0.99

∴ The person must arrive atleast 3.8 mins. early, so as to be 99% sure of seeing the start of the picture.

③ Arrivals at a telephone booth are considered to be poisson with an average time of 12 min. between one arrival & the next. The length of a phone call is assumed to be distributed exponentially with mean 4 min.

(i) Find the average no/. of persons waiting in the system. L_s

(ii) What is the probability that a person arriving at the booth will have to wait in the queue? $P(W>0)$

(iii) What is the prob/. that it will take him more than 10 min. altogether to wait for the phone & complete his call?

(iv) Estimate the fraction of the day when the phone will be in use.

(v) The telephone department will install a second booth, when convinced that an arrival has to wait on the average for atleast 3 min. for phone. By how much the flow of arrivals should increase in order to justify a second booth?

(vi) What is the average length of the queue, that forms from time to time?

Sol: Given: Telephone booth → Single server

Arrivals at a Telephone booth → Infinite capacity

∴ The given problem is $(M/M/1):(\infty/FIFO)$ model.

Mean arrival rate $\lambda = \frac{1}{12}$ per min.

Mean service rate $\mu = \frac{1}{4}$ per min.

$$\rho = \frac{\lambda}{\mu} = \frac{1/12}{1/4} = \frac{4}{12} = \frac{1}{3}$$

(i) Average no/. of persons waiting in the system:

$$L_s = \frac{\rho}{1-\rho} = \frac{1/3}{1-1/3} = \frac{1/3}{2/3} = \frac{1}{2}$$

(ii) Prob/. that the person arriving at the booth will have to wait in the queue

$$= P[W>0] = 1-P[W=0] = 1-P\,[\text{no customer in the system}]$$

$$= 1-P_0 = 1-(1-\rho) = \rho = \frac{1}{3}$$

(iii) A person takes more than 10 mins. to wait & complete his call:

$$P[W_s > t] = e^{-(\mu-\lambda)t}$$

$$P[W_s > 10] = e^{-\left(\frac{1}{4}-\frac{1}{12}\right)10} = e^{-5/3} = 0.1889$$

(iv) $P[\text{Phone in use}] = P[\text{Phone is busy}] = 1-P[\text{Phone is idle}]$

$$= 1-P_0 = 1-(1-\rho) = \rho = \frac{1}{3}$$

(v) The second phone will be installed if $E[W_q] \geq 3$

(v) $\frac{\lambda_1}{\mu(\mu-\lambda_1)} \geq 3$ where λ_1 is the required arrival rate.

(v) $\frac{\lambda_1}{\frac{1}{4}\left(\frac{1}{4}-\lambda_1\right)} \geq 3 \Rightarrow \lambda_1 \geq \frac{3}{4}\left(\frac{1}{4}-\lambda_1\right)$

$$\Rightarrow \lambda_1 \geq \frac{3}{16} - \frac{3\lambda_1}{4}$$

$$\Rightarrow \lambda_1 + \frac{3\lambda_1}{4} \geq \frac{3}{16}$$

$$\Rightarrow \frac{7\lambda_1}{4} \geq \frac{3}{16} \Rightarrow \lambda_1 \geq \frac{3}{16} \times \frac{4}{7} \Rightarrow \lambda_1 \geq \frac{3}{28}$$

Hence the arrival rate should increase by $\frac{3}{28} - \frac{1}{12} = \frac{1}{42}$ per min.

(vi) The average length of the queue $= \frac{\mu}{\mu-\lambda} = \frac{1/4}{1/4 - 1/12} = \frac{1/4}{2/12} = \frac{1}{4} \times 6 = 3/2 = 1.5$

④ In a railway marshalling yard, goods trains arrive at a rate of 30 trains per day. Assuming that the inter-arrival time follows an exponential distribution & the service time distribution is also exponential with an average 36 mins. Calculate the following: (i) The mean queue size.

(ii) The prob. that the queue size exceeds 10.

(iii) If the input of trains increases to an average of 33 per day what will be change in the above quantities?

Sol: Given: A railway marshalling yard → Single server

Goods trains → Infinite capacity

∴ The given problem is $(M/M/1):(\infty/FIFO)$ model.

Mean arrival rate $\lambda = 30$ trains per day $= \frac{30}{24} \times \frac{1}{60} = \frac{1}{48}$ per min.

Mean service rate $\mu = \frac{1}{36}$ per min.

$$\rho = \frac{\lambda}{\mu} = \frac{1/48}{1/36} = \frac{36}{48} = \frac{3}{4}$$

(i) Mean queue size: $L_q = \frac{\lambda^2}{\mu(\mu-\lambda)} = \frac{(1/48)^2}{1/36\,(1/36 - 1/48)} = \frac{1/2304}{1/36\,(1/144)} = \frac{36 \times 144}{2304}$

$= 9/4 = 2.25$ trains

(ii) $P(\text{Queue size exceeds } 10) = P(N_{\wedge}^{-1} > 10) = \rho^{11+1}$ $(\because P(N>n) = \rho^{n+1})$

$= \rho^{12} = \left(3/4\right)^{12} =$ ~~[illegible]~~ 0.0317

(iii) If the input of trains increases to an average of 33 per day.

Here $\lambda = 33$ trains per day $= \frac{33}{24} \times \frac{1}{60} = \frac{11}{480}$ per min.

$\mu = \frac{1}{36}$ per min.

$$\rho = \frac{\lambda}{\mu} = \frac{11/480}{1/36} = \frac{11 \times 36}{480} = \frac{33}{40}$$

$$L_s = \frac{\rho}{1-\rho} = \frac{33/40}{1-33/40} = \frac{33}{40} \times \frac{40}{7} = \frac{33}{7}$$

$L_q = L_s - \rho = \frac{33}{7} - \frac{33}{40} = 3.8893$ trains.

$\therefore$ Change in average queue size $= 3.8893 - 2.25 = 1.6393$ trains

Also, P(Queue size exceeds 10) $= P(N-1 > 10) = \rho^{11+1} = \rho^{12} = \left(\frac{33}{40}\right)^{12} =$ ~~0.1205~~ 0.0994

$\therefore$ Increment = ~~0.1205 − 0.0422 = 0.0783~~ $0.0994 - 0.0317 = 0.0677$.

Characteristics of Infinite capacity, multiple server Poisson queue:

Model II : $(M/M/s):(\infty/FIFO)$ model, when $\lambda_n = \lambda$, $\forall n$ $(\lambda < s\mu)$

Formulae: $\rho = \frac{\lambda}{s\mu}$

① $P_n = \frac{1}{s!\, s^{n-s}} \left(\frac{\lambda}{\mu}\right)^n P_0$ if $n \geq s$

② $P_0 = \left[\sum_{n=0}^{s-1} \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n + \frac{1}{s!\left(1-\frac{\lambda}{\mu s}\right)}\left(\frac{\lambda}{\mu}\right)^s\right]^{-1}$

③ $L_q = \frac{1}{s.s!} \frac{\left(\frac{\lambda}{\mu}\right)^{s+1}}{\left(1-\frac{\lambda}{\mu s}\right)^2} P_0$

④ $L_s = L_q + \frac{\lambda}{\mu} = \frac{1}{s.s!} \frac{\left(\frac{\lambda}{\mu}\right)^{s+1}}{\left(1-\frac{\lambda}{\mu s}\right)^2} P_0 + \frac{\lambda}{\mu}$

⑤ $W_s = \frac{1}{\lambda} L_s$

⑥ $W_q = \frac{1}{\lambda} L_q$

⑦ The average no/. of customers (in non-empty queues), who have to actually wait: $L_W = \frac{\rho}{1-\rho}$

⑧ The mean waiting time in the queue for those who actually wait: $W_n = \frac{1}{\mu s - \lambda}$

⑨ The probl. that an arrival has to wait = the probl. that there are s or more customers in the system: $P(W_s > 0) = P(N \geq s) = \frac{\left(\frac{\lambda}{\mu}\right)^s P_0}{s!(1-\rho)}$

⑩ $P(N > s+1) = \frac{\left(\frac{\lambda}{\mu}\right)^s}{s!} P_0 \left(\frac{\rho}{1-\rho}\right)$

Problems:

① There are 3 typists in an office. Each typist can type an average of 6 letters per hour. If letters arrive for being typed at the rate of 15 letters per hour. (i) What fraction of the time all the typists will be busy? $P(N \geq 3)$

(ii) What is the average no/. of letters waiting to be typed? L_q

(iii) What is the average time a letter has to spend for waiting & for being typed? W_s

(iv) What is the prob/. that a letter will take longer than 20 min. waiting to be typed & being typed? $P(W_s > t)$

Sol: Given 3 typists → Multiple server

Letters → Infinite capacity

Mean arrival rate $\lambda = 15$ per hour $= \frac{15}{60} = \frac{1}{4}$ per min.

Mean service rate $\mu = 6$ per hour $= \frac{6}{60} = \frac{1}{10}$ per min.

$$s = 3 \;;\quad \frac{\lambda}{\mu} = \frac{1/4}{1/10} = \frac{10}{4} = 5/2$$

$$\rho = \frac{\lambda}{\mu s} = \frac{1/4}{\frac{1}{10} \times 3} = \frac{1}{4} \times \frac{10}{3} = \frac{5}{6}$$

$$P_0 = \left[\sum_{n=0}^{s-1} \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n + \frac{\left(\frac{\lambda}{\mu}\right)^s}{s!(1-\rho)}\right]^{-1}$$

$$= \left[\sum_{n=0}^{2} \frac{1}{n!}\left(5/2\right)^n + \frac{(5/2)^3}{3!\,(1-5/6)}\right]^{-1}$$

$$= \left[1 + \frac{1}{1!}(5/2) + \frac{1}{2!}(5/2)^2 + \frac{\frac{125}{8}}{6(1/6)}\right]^{-1} = \left[1 + \frac{5}{2} + \frac{25}{8} + \frac{125}{8}\right]^{-1}$$

$$= \left[\frac{89}{4}\right]^{-1} = \frac{4}{89} = 0.0449$$

(i) $P[\text{all the typists are busy}] = P[N \geq 3]$

$$P[N \geq s] = \frac{\left(\frac{\lambda}{\mu}\right)^s P_0}{s!(1-\rho)}$$

$$P[N \geq 3] = \frac{(5/2)^3 (4/89)}{3!\,(1-5/6)} = \frac{\frac{125}{8} \times \frac{4}{89}}{6 \times 1/6} = \frac{125}{178} = 0.7022$$

(ii)
$$L_q = \frac{1}{s.s!} \frac{(\lambda/\mu)^{s+1}}{(1-\rho)^2} P_0 = \frac{1}{3.3!} \frac{(5/2)^4}{(1-5/6)^2}\left(\frac{4}{89}\right)$$

$$= \frac{1}{18} \cdot \frac{625/16}{1/36}\left(\frac{4}{89}\right) = \frac{1}{18} \times \frac{625}{16} \times 36 \times \frac{4}{89} = \frac{625}{178} = 3.5112$$

(iii) $W_s = \frac{1}{\lambda} L_s = \frac{1}{\lambda}\left[L_q + \frac{\lambda}{\mu}\right] = \frac{1}{1/4}\left[3.5112 + 5/2\right] = 24.0448$ mins.

(iv) $$P(W_s > t) = e^{-\mu t}\left[1 + \frac{\left(\frac{\lambda}{\mu}\right)^s \left[1 - e^{-\mu t\left(s-1-\frac{\lambda}{\mu}\right)}\right]}{s!\,(1-\rho)(s-1-\lambda/\mu)} P_0\right]$$

$$P(W_s > 20) = e^{-20/10}\left[1 + \frac{(5/2)^3\left(1 - e^{-2(3-1-5/2)}\right)}{3!\,(1-5/6)(3-1-5/2)} \cdot \frac{4}{89}\right]$$

$$= e^{-2}\left[1 + \frac{(125/8)(1-e^{1})}{6(1/6)(-0.5)} \cdot \frac{4}{89}\right] = e^{-2}\left[1 + \frac{125(1-e^{1})}{8 \times -0.5} \times \frac{4}{89}\right]$$

$= 0.4619$

② A petrol pump station has 4 pumps. The service times follow the exponential distribution with a mean of 6 mins. & cars arrive for service in a Poisson process at the rate of 30 cars per hour.

(i) What is the probability that an arrival would have to wait in line? $P(W>0)$

(ii) Find the average waiting time spent in the system & the average no. of cars in the system. W_s, L_s

(iii) For what % of time would a pump be idle on an average?

Sol: Given Petrol pumps → Multiple server

Cars → Infinite capacity

∴ The given problem is $(M/M/s):(\infty/FIFO)$ model.

Mean arrival rate $\lambda = 30$ per hour $= \frac{30}{60} = \frac{1}{2}$ per min.

Mean service rate $\mu = \frac{1}{6}$ per min.

$s = 4$

$$\rho = \frac{\lambda}{s\mu} = \frac{1/2}{4(1/6)} = \frac{1}{2} \times \frac{3}{2} = \frac{3}{4} \quad ; \quad \frac{\lambda}{\mu} = \frac{1/2}{1/6} = \frac{6}{2} = 3$$

$$P_0 = \left[\sum_{n=0}^{s-1} \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n + \frac{\left(\frac{\lambda}{\mu}\right)^s}{s!\,(1-\rho)}\right]^{-1} = \left[\sum_{n=0}^{3} \frac{1}{n!}(3)^n + \frac{(3)^4}{4!\,(1-3/4)}\right]^{-1}$$

$$= \left[1 + 3 + \frac{9}{2} + \frac{27}{6} + \frac{81}{24(1/4)}\right]^{-1} = \left[\frac{53}{2}\right]^{-1} = \frac{2}{53} = 0.0377$$

(i) $P[\text{an arrival has to wait}] = P[W > 0] = P[N \geq s]$

$$= \frac{\left(\frac{\lambda}{\mu}\right)^s}{s!\,(1-\rho)} P_0 = \frac{(3)^4}{4!\,(1-3/4)} \left(\frac{2}{53}\right)$$

$= 27/53 = 0.5094$

Scanned with CamScanner

(ii) $$L_q = \frac{1}{s.s!}\frac{\left(\frac{\lambda}{\mu}\right)^{s+1}}{\left(1-\frac{\lambda}{\mu s}\right)^2}P_0 = \frac{1}{4.4!}\frac{(3)^5}{(1-3/4)^2}\left(\frac{2}{53}\right)$$

$$= \frac{1}{4.4!}(3)^5(4)^2\left(\frac{2}{53}\right) = \frac{81}{53}$$

$$\therefore L_s = L_q + \frac{\lambda}{\mu} = \frac{81}{53} + 3 = \frac{240}{53} = 4.5283 \text{ cars}$$

$$\therefore W_s = \frac{1}{\lambda}L_s = \frac{1}{1/2}\times\frac{240}{53} = \frac{2\times 240}{53} = \frac{480}{53} = 9.0566 \text{ mins.}$$

(iii) The pumps are idle $\overset{P_0}{=} \rho = 3/4 = 3/4 = 0.75$

$\therefore$ 75% of time the pumps be idle.

③ A supermarket has 2 girls attending to sales at the counters. If the service time for each customer is exponential with mean 4 mins. & if people arrive in Poisson fashion at the rate of 10 per hour.

(i) What is the prob. that a customer has to wait for service? $P(W>0)$

(ii) What is the expected % of idle time for each girl? P_0

(iii) If the customer has to wait in the queue, what is the expected length of his waiting time? (non-empty)

Sol: Given 2 girls → multiple server

People → Infinite capacity

$\therefore$ The given problem is $(M/M/s):(\infty/FIFO)$ model.

Mean arrival rate $\lambda = 10$ per hour $= \frac{10}{60} = \frac{1}{6}$ per min.

Mean service rate $\mu = \frac{1}{4}$ per min.

$$s = 2$$

$$\frac{\lambda}{\mu} = \frac{1/6}{1/4} = \frac{4}{6} = 2/3 \quad ; \quad \rho = \frac{\lambda}{s\mu} = \frac{1}{2}\times\frac{2}{3} = 1/3$$

$$P_0 = \left[\sum_{n=0}^{s-1}\frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n + \frac{1}{s!(1-\rho)}\left(\frac{\lambda}{\mu}\right)^s\right]^{-1}$$

$$= \left[\sum_{n=0}^{1}\frac{1}{n!}(2/3)^n + \frac{1}{2!(1-1/3)}(2/3)^2\right]^{-1}$$

$$= \left[1 + \frac{2}{3} + \frac{3}{4}\left(\frac{4}{9}\right)\right]^{-1} = [2]^{-1} = \frac{1}{2}$$

(i) $$P[\text{a customer has to wait}] = P[W>0] = P[N \geq s] = \frac{\left(\frac{\lambda}{\mu}\right)^s}{s!(1-\rho)}P_0$$

$$= \frac{(2/3)^2}{2!(1-1/3)}\left(\frac{1}{2}\right) = \frac{4/9}{4/3}(1/2) = \frac{1}{6}$$

(ii) Idle time $= 1-\rho = 1-\frac{1}{3} = \frac{2}{3} = 0.6667$

$\therefore$ The expected % of idle time for each girl $= 66.67\%$.

(iii) $W_n = \frac{1}{\mu s-\lambda} = \frac{1}{\frac{1}{4}(2)-\frac{1}{6}} = \frac{1}{\frac{1}{2}-\frac{1}{6}} = \frac{6}{3-1} = \frac{6}{2} = 3$ mins.

④ A telephone exchange has 2 long distance operators. The telephone company finds that during the peak load, long distance calls arrive in a Poisson fashion at an average rate of 15 per hour. The length of service on these calls is approximately exponentially distributed with mean length 5 min.

(a) What is the probl. that a subscriber will have to wait for his long distance call during the peak hours of the day? $P[W>0]$

(b) If the subscribers will wait & are serviced in turn, what is the expected waiting time? W_q

Sol: Given 2 operators → multiple server

Calls → Infinite capacity

$\therefore$ The given problem is $(M/M/s):(\infty/FIFO)$ model.

Mean arrival rate $\lambda = 15$ per hour $= \frac{15}{60} = \frac{1}{4}$ per min.

Mean service rate $\mu = \frac{1}{5}$ per min. ; $s=2$

$\frac{\lambda}{\mu} = \frac{\frac{1}{4}}{\frac{1}{5}} = \frac{5}{4}$; $\rho = \frac{\lambda}{s\mu} = \frac{1}{2}\times\frac{5}{4} = \frac{5}{8}$

$$P_0 = \left[\sum_{n=0}^{s-1}\frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n + \frac{1}{s!(1-\rho)}\left(\frac{\lambda}{\mu}\right)^s\right]^{-1}$$

$$= \left[\sum_{n=0}^{1}\frac{1}{n!}\left(\frac{5}{4}\right)^n + \frac{1}{2!(1-\frac{5}{8})}\left(\frac{5}{4}\right)^2\right]^{-1}$$

$$= \left[1+\frac{5}{4}+\frac{8}{6}\left(\frac{25}{16}\right)\right]^{-1}$$

$$= \left[\frac{13}{3}\right]^{-1} = \frac{3}{13}$$

(a) $P[\text{a customer has to wait}] = P[W>0] = P[N\geq s] = \frac{\left(\frac{\lambda}{\mu}\right)^s}{s!(1-\rho)}P_0$

$$= \frac{\left(\frac{5}{4}\right)^2}{2!(1-\frac{5}{8})}\left(\frac{3}{13}\right) = \frac{25/16}{2\times 3/8}\left(\frac{3}{13}\right) = \frac{25}{16}\times\frac{4}{3}\times\frac{3}{13} = \frac{25}{52}$$

$= 0.4808$

(b) $L_q = \frac{1}{s.s!}\frac{\left(\lambda/\mu\right)^{s+1}}{(1-\rho)^2}P_0 = \frac{1}{2.2!}\frac{\left(\frac{5}{4}\right)^3}{(1-\frac{5}{8})^2}\times\frac{3}{13} = \frac{1}{4}\times\frac{125}{64}\times\frac{64}{9}\times\frac{3}{13}$

$L_q = \frac{125}{156}$

$\therefore W_q = \frac{1}{\lambda} L_q = \frac{1}{1/4} \times \frac{125}{156} = 4 \times \frac{125}{156} = \frac{125}{39} = 3.2051$ mins.

Characteristics of finite capacity, single server Poisson queue: model iii (M/M/1):(K/FIFO) model:

Formulae:

① $\rho = \frac{\lambda}{\mu}$, if $\lambda \neq \mu$

② $P_0 = \begin{cases} \frac{1-\rho}{1-\rho^{K+1}}, & \text{if } \lambda \neq \mu \\ \frac{1}{K+1}, & \text{if } \lambda = \mu \end{cases}$

③ $P_n = \begin{cases} \rho^n P_0 & \text{if } \lambda \neq \mu \\ \frac{1}{K+1} & \text{if } \lambda = \mu \end{cases}$

④ Effective arrival rate $= \lambda' = \mu(1-P_0) = \lambda(1-P_n)$; $L_s = \frac{K}{2}$ if $\lambda = \mu$

⑤ Little's formula: $L_s = \frac{\rho}{1-\rho} - \frac{(K+1)\rho^{K+1}}{1-\rho^{K+1}}$; $L_q = L_s - \frac{\lambda'}{\mu}$; $W_s = \frac{1}{\lambda'} L_s$;

$W_q = \frac{1}{\lambda'} L_q$

⑥ $P[\text{a customer turned away}] = P_K = \rho^K P_0$.

Problems:

① Patients arrive at a clinic according to Poisson distribution at a rate of 30 patients per hour. The waiting room does not accommodate more than 14 patients. Examination time per patient is exponential with mean rate of 20 per hour.

(i) Find the effective arrival rate at the clinic. λ'

(ii) What is the probl. that an arriving patient will not wait? P_0

(iii) What is the expected waiting time until a patient is discharged from the clinic? W_s

Sol: Given Clinic → Single server

Patients → Finite capacity

$\therefore$ The given problem is (M/M/1):(K/FIFO) model.

Mean arrival rate $\lambda = 30$ per hour $= \frac{30}{60} = \frac{1}{2}$ per min.

Mean service rate $\mu = 20$ per hour $= \frac{20}{60} = \frac{1}{3}$ per min.

$K = 14 + 1 = 15$

$$\therefore \rho = \frac{\lambda}{\mu} = \frac{1/2}{1/3} = \frac{3}{2}$$

$$P_0 = \frac{1-\rho}{1-\rho^{K+1}} \quad \text{if } \lambda \neq \mu$$

$$= \frac{1-3/2}{1-(3/2)^{15+1}} = \frac{-1/2}{-655.8408} = 0.00076$$

(i) Effective arrival rate $\lambda' = \mu(1-P_0) = \frac{1}{3}(1-0.00076) = 0.3331$ per min.

(ii) P(a patient will not wait) $= P_0 = 0.00076$

(iii) $$L_s = \frac{\rho}{1-\rho} - \frac{(K+1)\rho^{K+1}}{1-\rho^{K+1}} = \frac{3/2}{1-3/2} - \frac{16(3/2)^{16}}{1-(3/2)^{16}} = \frac{3/2}{-1/2} - \frac{16(3)^{16}}{2^{16}-3^{16}}$$

$$= -3 + 16.0244 = 13.0244$$

Expected waiting time $W_s = \frac{1}{\lambda'} L_s = \frac{1}{0.3331} \times 13.0244 = 39.1006$ mins.

② The one person barber shop can accommodate a maximum of 5 people at a time (4 waiting & 1 getting hair-cut). Customers arrive according to a Poisson distribution with mean 5/hr. The barber cuts hair at an average rate of 4/hr. (Exponential service rate). (i) What % of time is the barber idle? P_0
(ii) What fraction of the potential customers are turned away? $P[N=5]$
(iii) What is the expected no. of customers waiting for a hair-cut? L_q
(iv) How much time can a customer expect to spend in the barber shop? W_s

Sol: Given one person barber shop → Single server
Capacity → finite capacity

∴ The given problem is $(M/M/1):(K/FIFO)$ model.

Mean arrival rate $\lambda = 5$ per hour $= \frac{5}{60} = \frac{1}{12}$ per min.

Mean service rate $\mu = 4/hr = \frac{4}{60} = \frac{1}{15}$ per min.

$$K = 5 \; ; \; \rho = \frac{\lambda}{\mu} = \frac{1/12}{1/15} = \frac{15}{12} = 5/4$$

(i) $$P_0 = \frac{1-\rho}{1-\rho^{K+1}} = \frac{1-5/4}{1-(5/4)^6} = \frac{-1/4}{-2.8147} = 0.0888$$

∴ The percentage of time when the barber is idle = 8.88%.

(ii) P[a customer turned away] $= P[N=5] = P_5 = \rho^5 P_0$ $\quad (\because P_n = \rho^n P_0$ if $\lambda \neq \mu)$

$$= (5/4)^5 (0.0888) = 0.271$$

(iii) $\lambda' = \mu(1-P_0) = \frac{1}{15}(1-0.0888) = 0.0607$

$$L_s = \frac{\rho}{1-\rho} - \frac{(K+1)\rho^{K+1}}{1-\rho^{K+1}} = \frac{5/4}{1-5/4} - \frac{6(5/4)^6}{1-(5/4)^6} = -5 - \frac{6(5)^6}{4^6-5^6} = -5+8.1317$$

$$= 3.1317$$

$L_q = L_s - \frac{\lambda'}{\mu} = 3.1317 - \frac{0.0607}{1/15} = 2.2212$ customers.

(v) $W_s = \frac{L_s}{\lambda'} = \frac{3.1317}{0.0607} = 51.5931$ mins.

③ At a railway station, only one train is handled at a time. The railway yard is sufficient only for 2 trains to wait, while the other is given signal to leave the station. Trains arrive at the station at an average rate of 6 per hour & the railway station can handle them on an average of 6 per hour. Assuming Poisson arrivals & exponential service distribution, find the ~~probabilities~~ average for the nos/. of trains in the system L_s. Also find the average waiting time of a new train coming into the yard W_s. If the handling rate is doubled, how will the above results are modified?

Sol: Given One yard → Single server

Capacity → finite capacity

∴ The given problem is (M/M/1) : (K/FIFO) model.

Mean arrival rate $\lambda = 6$ per hour $= \frac{6}{60} = 1/10$ per min.

Mean service rate $\mu = 6$ per hour $= 1/10$ per min.

$K = 3$; $\rho = \frac{\lambda}{\mu} = \frac{1/10}{1/10} = 1$

$L_s = \frac{K}{2}$ if $\lambda = \mu = \frac{3}{2} = 1.5$ trains.

$P_0 = \frac{1}{K+1} = \frac{1}{3+1} = 1/4$; $\lambda' = \mu(1-P_0) = \frac{1}{10}(1 - 1/4) = \frac{1}{10} \times 3/4 = \frac{3}{40}$

$W_s = \frac{L_s}{\lambda'} = \frac{3/2}{3/40} = 20$ mins.

If the handling rate is doubled, $\lambda = 6$ per hour $= 1/10$ per min.

$\mu = 12/hr = \frac{12}{60} = 1/5$ per min. , $K = 3$

$\rho = \frac{\lambda}{\mu} = \frac{1/10}{1/5} = 1/2$.

$L_s = \frac{\rho}{1-\rho} - \frac{(K+1)\rho^{K+1}}{1-\rho^{K+1}} = \frac{1/2}{1-1/2} - \frac{4(1/2)^4}{1-(1/2)^4} = 1 - \frac{4}{2^4 - 1} = 0.7333$

$P_0 = \frac{1-\rho}{1-\rho^{K+1}} = \frac{1-1/2}{1-(1/2)^4} = \frac{1/2}{0.9375} = \frac{8}{15}$

$\lambda' = \mu(1-P_0) = \frac{1}{5}(1 - 8/15) = \frac{7}{75}$

$W_s = \frac{L_s}{\lambda'} = \frac{0.7333}{7/75} = 7.8568$ mins.

Characteristics of finite queue, multiple server Poisson queue model (IV):
(M/M/s): (K/FIFO) model:

① $\rho = \frac{\lambda}{s\mu}$

② $P_0 = \left[\sum_{n=0}^{s-1} \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n + \frac{1}{s!}\left(\frac{\lambda}{\mu}\right)^s \sum_{n=s}^{K} \rho^{n-s}\right]^{-1}$

③ $P_n = \frac{1}{s!}\frac{1}{s^{n-s}}\left(\frac{\lambda}{\mu}\right)^n P_0, \ s \le n \le K$; $P_n = \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n P_0, \ n \le s$; $P_n = 0, \ n > K$.

④ Effective arrival rate $\lambda' = \mu\left[s - \sum_{n=0}^{s-1}(s-n)P_n\right]$

⑤ Little's formula:

$$L_s = \frac{P_0}{s!}\left(\frac{\lambda}{\mu}\right)^s \left[\frac{\rho(1-\rho^{K-s})}{(1-\rho)^2} - \frac{(K-s)\rho^{K-s+1}}{1-\rho}\right] + \frac{\lambda'}{\mu}$$

$L_q = L_s - \frac{\lambda'}{\mu}$; $W_s = \frac{1}{\lambda'}L_s$; $W_q = \frac{L_q}{\lambda'}$

⑥ $P[n \ge s] = \frac{\mu(\lambda/\mu)^s}{(s-1)!(s\mu-\lambda)} P_0$

Problems:

① A car servicing station has 2 bays where service can be offered simultaneously. Because of space limitations, only 4 cars are accepted for servicing. The arrival pattern is Poisson with 12 cars per day. The service time in both the bays is exponentially distributed with $\mu = 8$ cars per day. Find the average no. of cars in the service station, the average no. of cars waiting for service & the average time a car spends in the system.

Sol: Given 2 bays → Multiple server

Capacity → Finite capacity

∴ The given problem is (M/M/s): (K/FIFO) model.

Here $s = 2$, $K = 4$

Mean arrival rate $\lambda = 12$ per day $= \frac{12}{24 \times 60} = \frac{1}{120}$ per min.

Mean service rate $\mu = 8$ per day $= \frac{8}{24 \times 60} = \frac{1}{180}$ per min.

$\frac{\lambda}{\mu} = \frac{1/120}{1/180} = \frac{180}{120} = \frac{3}{2}$; $\rho = \frac{\lambda}{s\mu} = \frac{1}{2} \times \frac{3}{2} = \frac{3}{4}$

$$P_0 = \left[\sum_{n=0}^{s-1}\frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n + \frac{1}{s!}\left(\frac{\lambda}{\mu}\right)^s \sum_{n=s}^{K} \rho^{n-s}\right]^{-1}$$

$$= \left[\sum_{n=0}^{1}\frac{1}{n!}(3/2)^n + \frac{1}{2!}(3/2)^2 \sum_{n=2}^{4}(3/4)^{n-2}\right]^{-1}$$

$$= \left[1 + \frac{3}{2} + \frac{9}{8}\left(1 + \frac{3}{4} + (3/4)^2\right)\right]^{-1} = \left[\frac{653}{128}\right]^{-1} = \frac{128}{653} = 0.196$$

$$\frac{\lambda'}{\mu} = s - \sum_{n=0}^{s-1}(s-n)P_n = 2 - \sum_{n=0}^{1}(2-n)P_n = 2 - [2P_0 + P_1]$$

$$= 2 - 2P_0 - P_1$$

$P_n = \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n P_0$ if $n \leq s$

$P_1 = \frac{3}{2} \times 0.196 = 0.294$; $\frac{\lambda'}{\mu} = 2 - 2(0.196) - 0.294 = 1.314$

$\lambda' = 1.314\mu = 1.314 \times \frac{1}{180} = 0.0073$

$$L_s = \frac{P_0}{s!}\left(\frac{\lambda}{\mu}\right)^s \left[\frac{\rho(1-\rho^{K-s})}{(1-\rho)^2} - \frac{(K-s)\rho^{K-s+1}}{1-\rho}\right] + \frac{\lambda'}{\mu}$$

$$= \frac{0.196}{2!}\left(\frac{3}{2}\right)^2 \left[\frac{3/4\left(1-(3/4)^{4-2}\right)}{(1-3/4)^2} - \frac{(4-2)(3/4)^{4-2+1}}{1-3/4}\right] + 1.314$$

$$= 0.2205\left[5.25 - 3.375\right] + 1.314 = 1.7274$$

$L_q = L_s - \frac{\lambda'}{\mu} = 1.7274 - 1.314 = 0.4134$

$W_s = \frac{L_s}{\lambda'} = \frac{1.7274}{0.0073} = 236.6301$ mins

② A 2 person barber shop has 5 chairs to accommodate waiting customers. Potential customers, who arrive when all 5 chairs are full leave without entering barber shop. Customers arrive at the average rate of 4 per hour & spend an average of 12 mins. in the barber's chair compute P_0, P_1, P_7, $E(N_q)$ & $E(W)$.

<u>Sol:</u> Given 2 person barber shop → multiple server

Capacity → finite capacity

∴ The given problem is $(M/M/s):(K/FIFO)$ model.

Mean arrival rate $\lambda = 4/hr = \frac{4}{60} = \frac{1}{15}$ per min.

Mean service rate $\mu = \frac{1}{12}$ per min. Here $s = 2$, $K = 5+2 = 7$

$\frac{\lambda}{\mu} = \frac{1/15}{1/12} = \frac{12}{15} = \frac{4}{5}$; $\rho = \frac{\lambda}{s\mu} = \frac{1}{2} \times \frac{4}{5} = 2/5$

$$P_0 = \left[\sum_{n=0}^{s-1}\frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n + \frac{1}{s!}\left(\frac{\lambda}{\mu}\right)^s \sum_{n=s}^{K}\rho^{n-s}\right]^{-1}$$

$$= \left[\sum_{n=0}^{1}\frac{1}{n!}\left(\frac{4}{5}\right)^n + \frac{1}{2!}\left(\frac{4}{5}\right)^2 \sum_{n=2}^{7}\left(2/5\right)^{n-2}\right]^{-1}$$

$$= \left[1 + \frac{4}{5} + 0.32\left(1 + \frac{2}{5} + \left(\frac{2}{5}\right)^2 + \left(\frac{2}{5}\right)^3 + \left(\frac{2}{5}\right)^4 + \left(\frac{2}{5}\right)^5\right)\right]^{-1} = (2.3311)^{-1}$$

$$= \frac{1}{2.3311} = 0.429$$

$$\frac{\lambda'}{\mu} = s - \sum_{n=0}^{s-1}(s-n)P_n = 2 - \sum_{n=0}^{1}(2-n)P_n = 2 - 2P_0 - P_1$$

$P_n = \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n P_0 \;, n \le s \quad (1<2)$

$P_1 = \frac{1}{1!}\left(\frac{4}{5}\right)^1 (0.429) = 0.3432$; $\therefore \frac{\lambda'}{\mu} = 2 - 2(0.429) - 0.3432 = 0.7988$

$P_n = \frac{1}{s!}\,\frac{1}{s^{n-s}}\left(\frac{\lambda}{\mu}\right)^n P_0 \;, s \le n \le k \quad (2<7\le 7)$

$P_7 = \frac{1}{2!}\,\frac{1}{2^{7-2}}\left(\frac{4}{5}\right)^7 (0.429) = \frac{1}{2}\,\frac{1}{2^5}\left(\frac{4}{5}\right)^7 (0.429) = 0.0014$

$E(N_q) = L_q = L_s - \frac{\lambda'}{\mu}$

$$L_s = \frac{P_0}{s!}\left(\frac{\lambda}{\mu}\right)^s \left[\frac{\rho(1-\rho^{k-s})}{(1-\rho)^2} - \frac{(k-s)\rho^{k-s+1}}{1-\rho}\right] + \frac{\lambda'}{\mu}$$

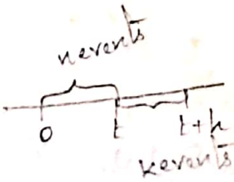
$$= \frac{0.429}{2!}\left(\frac{4}{5}\right)^2 \left[\frac{2/5\left(1-(2/5)^5\right)}{(1-2/5)^2} - \frac{5(2/5)^6}{1-2/5}\right] + 0.7988 = 0.9451 \text{ customer}$$

$E(N_q) = 0.9451 - 0.7988 = 0.1463$ customer

$E(W) = W_s = \frac{L_s}{\lambda'} = \frac{0.9451}{0.7988 \times 1/12} = 14.1978$ mins.

Birth and Death process:

The birth & death process is a special type of continuous time discrete state space, random process. WKT



$$P_n(h) = P[N(h)=k \,/\, N(t)=n] = \begin{cases} \lambda_n h + o(h) & , k=1 \\ o(h) & , k \ge 2 \\ 1-\lambda_n h + o(h) & , k=0 \end{cases}$$

This holds for all $n \ge 0$. Hence λ_n may or may not be equal to zero. When λ is a non-negative integer, this implies that there can only be an increase by k (ie) only births are considered possible. Suppose there could also be decrease by k. Let us assume,

$$q_n(h) = P[N(h)=k \,/\, N(t)=n] = \begin{cases} \mu_n h + o(h) & , k=1 \\ o(h) & , k \ge 2 \\ 1-\mu_n h + o(h) & , k=0 \end{cases} \quad \text{where } n \ge k \;\&\; \mu_0 = 0.$$

$P_n(h)$ & $q_n(h)$ are known as birth & death process. Through a birth there is an increase by one & through a death there is a decrease by one in the no/. of individuals. The probability of more than one birth or more than one death in an interval of length $o(h)$.

Equation of birth & death process:

Let $N(t)$ denotes the total no/. of individuals at 't' starting from $t=0$. Consider the interval 0 to $t+h$. Suppose that this is split up into 2 periods

0 to t & t to t+h. The event $\{N(t+h)=n, n\geq 1\}$ having the probability $P_n(t+h)$ can occur in a no. of mutually exclusive ways. These would include events involving more than one birth μ or more than one death between t & t+h. Such an event is o(h). There will remain four other events to be considered.

A_{ij} : $(n-i+j)$ individuals by epoch t, i-birth & j-death between t & t+h, $i,j=0,1$.

Let $P_n(t)=P[N(t)=n]$, then

$$P(A_{00})=P_n(t)\left[1-\lambda_n h+o(h)\right]\left[1-\mu_n h+o(h)\right]$$
$$=P_n(t)\left[1-(\lambda_n+\mu_n)h+o(h)\right]$$

$$P(A_{10})=P_{n-1}(t)\left[\lambda_{n-1}h+o(h)\right]\left[1-\mu_{n-1}h+o(h)\right]=P_{n-1}(t)\left[\lambda_{n-1}h+o(h)\right]$$

$$P(A_{01})=P_{n+1}(t)\left[1-\lambda_{n+1}h+o(h)\right]\left[\mu_{n+1}h+o(h)\right]=P_{n+1}(t)\left[\mu_{n+1}h+o(h)\right]$$

$$P(A_{11})=P_n(t)\left[\lambda_n h+o(h)\right]\left[\mu_n h+o(h)\right]=o(h)$$

∴ For $n\geq 1$, $P_n(t+h)=P_n(t)\left[1-(\lambda_n+\mu_n)h\right]+P_{n-1}(t)\left[\lambda_{n-1}h\right]+P_{n+1}(t)\mu_{n+1}h+o(h)$

$$\frac{P_n(t+h)-P_n(t)}{h}=-(\lambda_n+\mu_n)P_n(t)+P_{n-1}(t)\lambda_{n-1}+P_{n+1}(t)\mu_{n+1}+\frac{o(h)}{h}$$

As $h\to 0$,

$$\lim_{h\to 0}\frac{P_n(t+h)-P_n(t)}{h}=-(\lambda_n+\mu_n)P_n(t)+P_{n-1}(t)\lambda_{n-1}+P_{n+1}(t)\mu_{n+1}+\lim_{h\to 0}\frac{o(h)}{h}$$

$$P_n'(t)=-(\lambda_n+\mu_n)P_n(t)+P_{n-1}(t)\lambda_{n-1}+P_{n+1}(t)\mu_{n+1} \; ① \quad \left(\because \lim_{h\to 0}\frac{o(h)}{h}=0\right)$$

For $n=0$, $P_0(t+h)=P_0(t)\left[1-(\lambda_0+\mu_0)h\right]+P_1(t)\mu_1 h+o(h)$

$$\frac{P_0(t+h)-P_0(t)}{h}=-(\lambda_0+\mu_0)P_0(t)+P_1(t)\mu_1+\frac{o(h)}{h}$$

As $h\to 0$, $P_0'(t)=-(\lambda_0+\mu_0)P_0(t)+P_1(t)\mu_1$

$$\Rightarrow P_0'(t)=-\lambda_0 P_0(t)+\mu_1 P_1(t) \quad ② \quad (\because \mu_0=0)$$

If at epoch $t=0$ there were i (≥ 0) individuals, the initial condition

$$P_n(0)=0,\ n\neq 1,\ P_1(0)=1$$

∴ Equns. ① & ② are known as birth & death process.

ADVANCED QUEUEING MODELS

Derivation of Pollaczek - Khintchine (P-K) formula:

Let T represents the time when the j^{th} customer departs & (T+t) represents the time when the next $(j+1)^{th}$ customer departs.

Let $f(t) \rightarrow$ prob/. of service time distribution with mean E(t) & variance.

$n \rightarrow$ no/. of customers in the system just after a customer departs.

$t \rightarrow$ time to serve the customer following the one already departed.

$k \rightarrow$ no/. of new arrivals during t

$n' \rightarrow$ no/. of customers left behind the next departing customer.

The notations j, j+1, ... do not necessarily mean that customers all introduced into service on FCFS discipline. Rather, it identifies the different customers departing from the system. Thus the result of this model can be applied to any one of the 3 service disciplines FCFS, FCLS & SIRO.

	Queue	Service	Departure
T	n-1	$(j+1)^{th}$	j^{th}
T+t	n-2	$(j+2)^{th}$	$(j+1)^{th}$

The system is observed only just after a service departure has been completed. Such instant of time defines the regeneration point. If n customers are in the system at time T initially then at the next moment (T+t) the no/. n in the system is given by, $n' = \begin{cases} k, & \text{if } n=0 \\ (n-1)+k, & \text{if } n>0 \end{cases}$ where k=0,1,2,... is the no/. of arrivals during the service time.

Alternatively if $\delta = \begin{cases} 1, & \text{if } n=0 \\ 0, & \text{if } n>0 \end{cases}$

$$n' = n-1+\delta+k \text{ , then}$$

$$E(n') = E(n-1+\delta+k) = E(n)-1+E(\delta)+E(k)$$

Since E(n) = E(n') in the steady-state, $E(\delta) = 1-E(k)$

Also, $(n')^2 = [n+(k-1)+\delta]^2 = n^2+(k-1)^2+\delta^2+2n(k-1)+2\delta(k-1)+2n\delta$

Since δ can take values 0 & 1, so $\delta^2 = \delta$ & $n\delta = 0$

$$\therefore (n')^2 = n^2+k^2+1-2k+\delta+2nk-2n+2\delta k-2\delta$$

$$= n^2+k^2+1-2k-\delta-2n+2nk+2\delta k$$

$$= n^2+k^2+2n(k-1)+\delta(2k-1)-2k+1$$

$$\Rightarrow -2n(k-1) = n^2-n'^2+k^2+\delta(2k-1)-2k+1$$

$$\Rightarrow 2n(1-k) = n^2 - n'^2 + k^2 + \delta(2k-1) - 2k+1$$

$$\therefore E[2n(1-k)] = E(n^2) - E(n'^2) + E(k^2) + E(\delta)E(2k-1) - 2E(k) + E(1)$$

$$\Rightarrow 2E(n)E(1-k) = E(n^2) - E(n'^2) + E(k^2) + E(\delta)[2E(k)-1] - 2E(k)+1$$

$$\Rightarrow 2E(n)[1-E(k)] = E(n^2) - E(n'^2) + E(k^2) + E(\delta)[2E(k)-1] - 2E(k)+1$$

$$= E(k^2) + E(\delta)[2E(k)-1] - 2E(k)+1 \quad (\because E(n^2) = E(n'^2))$$

$$\Rightarrow E(n) = \frac{E(k^2)+E(\delta)[2E(k)-1]-2E(k)+1}{2[1-E(k)]}$$

Substituting $E(\delta)=1-E(k)$

$$\therefore E(n) = \frac{E(k^2)+[1-E(k)][2E(k)-1]-2E(k)+1}{2[1-E(k)]}$$

$$= \frac{E(k^2)+2E(k)-1-2E^2(k)+E(k)-2E(k)+1}{2[1-E(k)]}$$

$$= \frac{E(k^2)+E(k)-2E^2(k)}{2[1-E(k)]}$$

Now, in order to determine $E(n)$ the values of $E(k)$ & $E(k^2)$ are to be computed. Since the arrivals in time t follow the Poisson distribution,

$$E[k/t] = \lambda t \ , \quad E[k^2/t] = (\lambda t)^2 + \lambda t$$

$$V(k/t) = E[k^2/t] - [E(k/t)]^2$$

Hence $E(k) = \int_0^\infty E[k/t]\, f(t)\,dt = \int_0^\infty \lambda t f(t)\,dt = \lambda \int_0^\infty t f(t)\,dt = \lambda E(t)$

Also $E(k^2) = \int_0^\infty E[k^2/t]\, f(t)\,dt = \int_0^\infty ((\lambda t)^2 + \lambda t) f(t)\,dt$

$$= \lambda^2 \int_0^\infty t^2 f(t)\,dt + \lambda \int_0^\infty t f(t)\,dt = \lambda^2 E(t^2) + \lambda E(t)$$

$$\therefore E(n) = \frac{E(k^2)+E(k)-2E^2(k)}{2[1-E(k)]}$$

$$= \frac{\lambda^2 E(t^2)+\lambda E(t)+\lambda E(t)-2[\lambda^2 E^2(t)]}{2[1-\lambda E(t)]}$$

$$= \frac{\lambda^2 E(t^2)+2\lambda E(t)-2\lambda^2 E^2(t)}{2[1-\lambda E(t)]} = \frac{\lambda^2 E(t^2)+2\lambda E(t)-\lambda^2 E^2(t)-\lambda^2 E^2(t)}{2[1-\lambda E(t)]}$$

$$= \frac{2\lambda E(t)-\lambda^2 E^2(t)+\lambda^2[E(t^2)-E^2(t)]}{2[1-\lambda E(t)]}$$

$$= \frac{2\lambda E(t) - \lambda^2 E^2(t) + \lambda^2 V(t)}{2[1-\lambda E(t)]}$$

$$= \frac{2\lambda E(t) - 2\lambda^2 E^2(t) + \lambda^2 E^2(t) + \lambda^2 V(t)}{2[1-\lambda E(t)]}$$

$$= \frac{2\lambda E(t)[1-\lambda E(t)] + \lambda^2 E^2(t) + \lambda^2 V(t)}{2[1-\lambda E(t)]}$$

$$= \lambda E(t) + \frac{\lambda^2 [V(t) + E^2(t)]}{2[1-\lambda E(t)]}$$

$$\therefore L_s = E_n = E(n) = \lambda E(t) + \frac{\lambda^2 [V(t) + E^2(t)]}{2[1-\lambda E(t)]}$$

This is called as Pollaczek-Khintchine (P-K) formula.

Formulae:

① The average no. of customers in the system $L_s = \frac{\lambda^2\sigma^2 + \rho^2}{2(1-\rho)} + \rho$

② The average queue length $L_q = \frac{\lambda^2\sigma^2 + \rho^2}{2(1-\rho)}$

③ The average waiting time of a customer in the queue $W_q = \frac{\lambda^2\sigma^2 + \rho^2}{2\lambda(1-\rho)}$

④ The average waiting time that a customer spends in the system

$$W_s = \frac{\lambda^2\sigma^2 + \rho^2}{2\lambda(1-\rho)} + \frac{1}{\mu}$$

⑤ Little's formula: $L_q = L_s - \rho$; $W_s = \frac{L_s}{\lambda}$; $W_q = \frac{L_q}{\lambda}$.

Problems:

① In a heavy machine shop, the overhead crane is 75% utilised. Time study observations gave the average slinging time as 10.5 mins with a S.D. of 8.8 mins. What is the average calling rate for the services of the crane & what is the average delay in getting service? If the average service time is cut to 8 mins, with a S.D. of 6 mins, how much reduction will occur, on average, in the delay of getting served?

Sol: Given Utilisation rate $= 75\% = \frac{75}{100} = 3/4$ (ie) $\rho = 3/4$

Mean service rate $\mu = \frac{1}{10.5}$ per min.

WKT $\rho = \frac{\lambda}{\mu} \Rightarrow \lambda = \rho\mu = \left(3/4\right)\left(\frac{1}{10.5}\right) = 0.0714 = 1/14$. Here $\sigma = 8.8$

Average delay in getting service

$$W_q = \frac{\lambda^2\sigma^2+\rho^2}{2\lambda(1-\rho)} = \frac{(1/14)^2(8.8)^2+(3/4)^2}{2(1/14)(1-3/4)} = 26.8129 \text{ mins.}$$

The average service time is cut to 8 mins then $\mu = \frac{1}{8}$ per min

$\lambda = \frac{1}{14}$ per min. ; $\rho = \frac{\lambda}{\mu} = \frac{1/14}{1/8} = \frac{8}{14} = 4/7$; $\sigma = 6$ mins.

Average delay in getting service

$$W_q = \frac{\lambda^2\sigma^2+\rho^2}{2\lambda(1-\rho)} = \frac{(1/14)^2(6)^2+(4/7)^2}{2(1/14)(1-4/7)} = 8.3333 \text{ mins.}$$

∴ The average waiting time has a reduction of $26.8129 - 8.3333 = 18.4796$ mins.

② An automatic car wash facility operates with only one bay. Cars arrive according to a Poisson distribution with a mean of 4 cars/hr, & may wait in the facility's parking lot if the bay is busy. Find L_s, L_q, W_s, W_q if the service time,
(a) is constant & equal to 10 mins (b) follows uniform distribution between 8 & 12 mins.
(c) follows normal distribution with mean 12 mins. & S.D. 3 mins.
(d) follows a discrete distribution with values 4, 8 & 15 mins. with corresponding probabilities 0.2, 0.6 & 0.2.

Sol: Given $\lambda = 4$ per hr $= \frac{4}{60} = \frac{1}{15}$ per min.

(a) $\mu = \frac{1}{10}$ per min. Since the service time is constant, variance $\sigma^2 = 0$.

$$\rho = \frac{\lambda}{\mu} = \frac{1/15}{1/10} = \frac{10}{15} = 2/3$$

By Pollaczek-Khintchine formula

$$L_s = \frac{\lambda^2\sigma^2+\rho^2}{2(1-\rho)} + \rho = \frac{(1/15)^2(0)+(2/3)^2}{2(1-2/3)} + \frac{2}{3} = \frac{4/9}{2/3} + 2/3 = \frac{4}{3}$$

$L_q = L_s - \rho = \frac{4}{3} - \frac{2}{3} = 2/3$; $W_s = \frac{L_s}{\lambda} = \frac{4/3}{1/15} = \frac{4}{3} \times 15 = 20$ mins

$W_q = \frac{L_q}{\lambda} = \frac{2/3}{1/15} = \frac{2}{3} \times 15 = 10$ mins.

(b) When service time follows uniform distribution between 8 & 12. Here $a = 8$, $b = 12$

$$\mu = \frac{1}{\frac{a+b}{2}} = \frac{1}{8+12} \times 2 = \frac{2}{20} = \frac{1}{10} \text{ per min.}$$

$\sigma^2 = \frac{(b-a)^2}{12} = \frac{(12-8)^2}{12} = \frac{16}{12} = 4/3$; $\rho = \frac{\lambda}{\mu} = \frac{1/15}{1/10} = \frac{10}{15} = 2/3$

$$L_s = \frac{\lambda^2\sigma^2+\rho^2}{2(1-\rho)} + \rho = \frac{(1/15)^2(4/3)^2+(2/3)^2}{2(1-2/3)} + 2/3 = 1.3452$$

$L_q = L_s - \rho = 1.3452 - 2/3 = 0.6785$; $W_s = \frac{L_s}{\lambda} = 1.3452 \times 15 = 20.178$ mins.

$W_q = \frac{L_q}{\lambda} = 0.6785 \times 15 = 10.1775$ mins.

(c) When service time follows normal distribution with $\mu = \frac{1}{12}$ per min. & $\sigma = 3$ mins.

$$\rho = \frac{\lambda}{\mu} = \frac{1/15}{1/12} = \frac{12}{15} = \frac{4}{5}$$

$$L_s = \frac{\lambda^2\sigma^2 + \rho^2}{2(1-\rho)} + \rho = \frac{(1/15)^2(3)^2 + (4/5)^2}{2(1-4/5)} + \frac{4}{5} = 2.5$$

$L_q = L_s - \rho = 2.5 - \frac{4}{5} = 1.7$; $W_s = \frac{L_s}{\lambda} = \frac{2.5}{1/15} = 2.5 \times 15 = 37.5$ mins.

$W_q = \frac{L_q}{\lambda} = 1.7 \times 15 = 25.5$ mins.

(d)

t:	4	8	15
$p(t)$:	0.2	0.6	0.2

$E(T) = \sum t\,p(t) = 8.6$; $E(T^2) = \sum t^2 p(t) = 86.6$

$Var(T) = E(T^2) - [E(T)]^2 = 86.6 - (8.6)^2 = 12.64 = \sigma^2$

$\therefore \mu = \frac{1}{8.6}$; $\sigma^2 = 12.64$; $\rho = \frac{\lambda}{\mu} = \frac{1/15}{1/8.6} = \frac{8.6}{15} = 0.5733$

$$L_s = \frac{\lambda^2\sigma^2 + \rho^2}{2(1-\rho)} + \rho = \frac{(1/15)^2(12.64) + (0.5733)^2}{2(1-0.5733)} + 0.5733 = 1.0243$$

$L_q = L_s - \rho = 1.0243 - 0.5733 = 0.451$; $W_s = \frac{L_s}{\lambda} = 1.0243 \times 15 = 15.3645$ mins.

$W_q = \frac{L_q}{\lambda} = 0.451 \times 15 = 6.765$ mins.

Series Queues:

queues are not allowed

① For a 2-stage (service point) sequential queue model with blockage, compute the average no. of customers in the system & the average time that a customer has to spend in the system, if $\lambda = 1$, $\mu_1 = 2$ & $\mu_2 = 1$.

Sol: Given $\lambda = 1$, $\mu_1 = 2$, $\mu_2 = 1$

The balanced eqns. are

State	Rate that the process leaves	
$(0,0)$	$\lambda P_{00} = \mu_2 P_{01}$	①
$(1,0)$	$\mu_1 P_{10} = \lambda P_{00} + \mu_2 P_{11}$	②
$(0,1)$	$(\lambda + \mu_2) P_{01} = \mu_1 P_{10} + \mu_2 P_{b1}$	③
$(1,1)$	$(\mu_1 + \mu_2) P_{11} = \lambda P_{01}$	④
$(b,1)$	$\mu_2 P_{b1} = \mu_1 P_{11}$	⑤

$P_{00} + P_{10} + P_{01} + P_{11} + P_{b1} = 1$ — ⑥

① $\Rightarrow P_{00} = P_{01}$ — ⑦

② $\Rightarrow 2P_{10} = P_{00} + P_{11}$ — ⑧

$③ \Rightarrow 2P_{01} = 2P_{10} + P_{b1}$ —⑨

$④ \Rightarrow 3P_{11} = P_{01}$ —⑩

$⑤ \Rightarrow P_{b1} = 2P_{11}$ —⑪

$⑦ \,\&\, ⑩ \Rightarrow P_{00} = P_{01} = 3P_{11}$ —⑫

$⑫ \,\&\, ⑪ \Rightarrow P_{00} = P_{01} = 3P_{11} = \frac{3}{2}P_{b1}$ —⑬

$⑧ \Rightarrow 2P_{10} = P_{00} + \frac{1}{3}P_{00} = \frac{4}{3}P_{00} \Rightarrow P_{10} = \frac{2}{3}P_{00}$ —⑭

P[an arriving customer enters the system] $= P_{00} + P_{01}$

Effective arrival rate $= \lambda'$
$= \lambda(P_{00}+P_{01})$

$\therefore ⑥ \Rightarrow P_{00} + P_{10} + P_{01} + P_{11} + P_{b1} = 1$

$$P_{00} + \frac{2}{3}P_{00} + P_{00} + \frac{1}{3}P_{00} + \frac{2}{3}P_{00} = 1$$

$$\Rightarrow P_{00}\left(1+\frac{2}{3}+1+\frac{1}{3}+\frac{2}{3}\right) = 1 \Rightarrow P_{00}(11/3) = 1 \Rightarrow P_{00} = \frac{3}{11}$$

$$\therefore P_{01} = \frac{3}{11} \;;\; P_{11} = \frac{1}{3}P_{00} = \frac{1}{3}\times\frac{3}{11} = \frac{1}{11} \;;\; P_{b1} = \frac{2}{3}P_{00} = \frac{2}{3}\times\frac{3}{11} = \frac{2}{11}$$

$$P_{10} = \frac{2}{3}P_{00} = \frac{2}{3}\times\frac{3}{11} = \frac{2}{11}$$

$$L_s = P_{01} + P_{10} + 2(P_{11}+P_{b1}) = \frac{3}{11} + \frac{2}{11} + 2\left(\frac{1}{11}+\frac{2}{11}\right) = 1$$

$$W_s = \frac{L_s}{\lambda(P_{00}+P_{01})} = \frac{1}{1(3/11 + 3/11)} = \frac{11}{6}$$

② In a big factory, there are a large no. of operating machines & two sequential repair shops, which do the service of the damaged machines exponentially with respective rates of 1/hour & 2/hour. If the cumulative failure rate of all the machines in the factory is 0.5/hour, find (i) the prob. that both repair shops are idle, (ii) the average no. of machines in the service station of the factory & (iii) the average repair time of a machine.

Sol: Given $\lambda = 0.5/hr = \frac{0.5}{60}$ per min. $= \frac{1}{120}$ per min.

$\mu_1 = 1$ per hour $= \frac{1}{60}$ per min., $\mu_2 = 2$ per hour $= \frac{2}{60} = \frac{1}{30}$ per min.

The situation in this problem is comparable with 2-stage tandem queue with single server at each state.

$$P(n,m) = \left(\frac{\lambda}{\mu_1}\right)^n\left(1-\frac{\lambda}{\mu_1}\right)\left(\frac{\lambda}{\mu_2}\right)^m\left(1-\frac{\lambda}{\mu_2}\right)$$

(i) P(both the service stations are idle) $= P(0,0) = \left(\frac{\lambda}{\mu_1}\right)^0\left(1-\frac{\lambda}{\mu_1}\right)\left(\frac{\lambda}{\mu_2}\right)^0\left(1-\frac{\lambda}{\mu_2}\right)$

$$= \left(\frac{1/2}{1}\right)^0\left(1-\frac{1/2}{1}\right)\left(\frac{1/2}{2}\right)^0\left(1-\frac{1/2}{2}\right) = 1/2 \times 3/4 = 3/8$$

(ii) The average no. of machines in the service station of the factory

$$L_s = \frac{\lambda}{\mu_1-\lambda} + \frac{\lambda}{\mu_2-\lambda} = \frac{1/2}{1-1/2} + \frac{1/2}{2-1/2} = \frac{1/2}{1/2} + \frac{1/2}{3/2} = 1 + 1/3 = 4/3$$

(iii) the average repair time of a machine $W_s = \frac{L_s}{\lambda}$ (M/M/1):(∞/FIFO) model

$$W_s = \frac{1}{\mu_1 - \lambda} + \frac{1}{\mu_2 - \lambda} = \frac{1}{1 - 1/2} + \frac{1}{2 - 1/2} = \frac{1}{1/2} + \frac{1}{3/2} = 2 + \frac{2}{3} = 8/3$$

③ In a departmental store, there are 2 sections, namely, grocery section & perishable section. Customers from outside arrive at the G-section according to a Poisson process at a mean rate of 10/hour & they reach the P-section at a mean rate of 2/hour. The service times at both the sections are exponentially distributed with parameters 15 & 12 respectively. On finishing the job in the G-section, a customer is equally likely to go the P-section or to leave the store, whereas a customer on finishing his job in the P-section will go to the G-section with prob. 0.25 & leave the store otherwise. Assuming that there is only one salesman in each section, find the prob. that there are 3 customers in the G-section & 2 customers in the P-section. Find also the average no. of customers in the store & the average waiting time of a customer in the store.

Sol: Given $r_1 = 10$, $r_2 = 2$, $\lambda = r_1 + r_2 = 12$, $\mu_1 = 15$, $\mu_2 = 12$

$P_{11} = 0$, $P_{12} = 1/2$, $P_{21} = 0.25 = 1/4$, $P_{22} = 0$

The Jackson's flow balance eqns. are

$$\lambda_j = r_j + \sum_{i=1}^{2} \lambda_i P_{ij}, \quad j = 1, 2$$

For j=1, we get

$$\lambda_1 = r_1 + \sum_{i=1}^{2} \lambda_i P_{i1} = r_1 + \lambda_1 P_{11} + \lambda_2 P_{21} = 10 + \lambda_1(0) + \lambda_2(1/4)$$

$$\Rightarrow \lambda_1 = 10 + \frac{\lambda_2}{4} \quad \text{—①} \Rightarrow 4\lambda_1 = 40 + \lambda_2$$

For j=2, we get

$$\lambda_2 = r_2 + \sum_{i=1}^{2} \lambda_i P_{i2} = r_2 + \lambda_1 P_{12} + \lambda_2 P_{22} = 2 + \lambda_1(1/2) + \lambda_2(0)$$

$$\Rightarrow \lambda_2 = 2 + \frac{\lambda_1}{2} \quad \text{—②} \Rightarrow 2\lambda_2 = 4 + \lambda_1$$

① ⇒ $\cancel{4\lambda_1} - \lambda_2 = 40$

② × 4 ⇒ $\cancel{-4\lambda_1} + 8\lambda_2 = 16$

$7\lambda_2 = 56 \Rightarrow \lambda_2 = 8$

$\therefore \lambda_1 = 2\lambda_2 - 4 = 2(8) - 4 = 16 - 4 = 12$

Given $\mu_1=15$, $\mu_2=12$, $\lambda_1=12$, $\lambda_2=8$

$$P(n,m)=\left(\frac{\lambda_1}{\mu_1}\right)^n\left(1-\frac{\lambda_1}{\mu_1}\right)\left(\frac{\lambda_2}{\mu_2}\right)^m\left(1-\frac{\lambda_2}{\mu_2}\right)$$

$$P(3,2)=\left(\frac{12}{15}\right)^3\left(1-\frac{12}{15}\right)\left(\frac{8}{12}\right)^2\left(1-\frac{8}{12}\right)=\left(\frac{12}{15}\right)^3\left(\frac{3}{15}\right)\left(\frac{8}{12}\right)^2\left(\frac{4}{12}\right)$$

$$=\left(\frac{4}{5}\right)^3\left(\frac{1}{5}\right)\left(\frac{2}{3}\right)^2\left(\frac{1}{3}\right)=0.0152$$

$$L_s=\frac{\lambda_1}{\mu_1-\lambda_1}+\frac{\lambda_2}{\mu_2-\lambda_2}=\frac{12}{15-12}+\frac{8}{12-8}=\frac{12}{3}+\frac{8}{4}=4+2=6$$

$$W_s=\frac{L_s}{\lambda}=\frac{6}{12}=\frac{1}{2}\text{ hours}=\frac{1}{2}\times 60=30\text{mins.}$$

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④ In a network of 3 service stations 1,2,3 customers arrive 1,2,3 from outside, in accordance with Poisson process having rates 5,10,15 respectively. The service times at the 3 stations are exponential with respective rates 10,50,100. A customer completing service at station 1 is equally like to (i) go to station (2), (ii) go to station 3 & (iii) leave the system. A customer departing from service at station 2 always goes to station 3. A departure from service at station 3 is equally like to go to station 2 or leave the system.

(a) What is the average no. of customers in the system consisting of all the three stations?

(b) What is the average time a customer spends in the system?

Sol: Given $r_1=5$, $r_2=10$, $r_3=15$, $\lambda=r_1+r_2+r_3=30$

$\mu_1=10$, $\mu_2=50$, $\mu_3=100$.

Given $P_{11}=0$, $P_{12}=\frac{1}{3}$, $P_{13}=\frac{1}{3}$; $P_{21}=0$, $P_{22}=0$, $P_{23}=1$

$P_{31}=0$, $P_{32}=1/2$, $P_{33}=0$

The Jackson's flow balance equs/. are

$$\lambda_j=r_j+\sum_{i=1}^{3}\lambda_i P_{ij},\quad j=1,2,3$$

For $j=1$, we get

$$\lambda_1=r_1+\sum_{i=1}^{3}\lambda_i P_{i1}=r_1+\lambda_1P_{11}+\lambda_2P_{21}+\lambda_3P_{31}=5+\lambda_1(0)+\lambda_2(0)+\lambda_3(0)$$

$\lambda_1=5$

For $j=2$, we get

$$\lambda_2=r_2+\sum_{i=1}^{3}\lambda_i P_{i2}=r_2+\lambda_1P_{12}+\lambda_2P_{22}+\lambda_3P_{32}=10+\lambda_1(1/3)+\lambda_2(0)+\lambda_3(1/2)$$

$$\lambda_2=10+\frac{5}{3}+\frac{\lambda_3}{2}\Rightarrow\lambda_2=\frac{35}{3}+\frac{\lambda_3}{2}\Rightarrow 6\lambda_2=70+3\lambda_3 \quad —①$$

∴ For $j=3$, we get

$$\lambda_3 = r_3 + \sum_{i=1}^{3} \lambda_i P_{i3} = r_3 + \lambda_1 P_{13} + \lambda_2 P_{23} + \lambda_3 P_{33} = 15 + \lambda_1\left(\tfrac{1}{3}\right) + \lambda_2(1) + \lambda_3(0)$$

$$\lambda_3 = 15 + \frac{5}{3} + \lambda_2 \Rightarrow \lambda_3 = \frac{50}{3} + \lambda_2 \Rightarrow 3\lambda_3 = 50 + 3\lambda_2 \quad — ②$$

$$① \Rightarrow 6\lambda_2 = 70 + 3\lambda_3 = 70 + 50 + 3\lambda_2 \quad (\because \text{by } ②)$$

$$\Rightarrow 3\lambda_2 = 120 \Rightarrow \lambda_2 = 40$$

$$\therefore ② \Rightarrow 3\lambda_3 = 50 + 3(40) = 50 + 120 = 170$$

$$\Rightarrow \lambda_3 = \frac{170}{3}.$$

(a) $$L_s = \frac{\lambda_1}{\mu_1 - \lambda_1} + \frac{\lambda_2}{\mu_2 - \lambda_2} + \frac{\lambda_3}{\mu_3 - \lambda_3} = \frac{5}{10-5} + \frac{40}{50-40} + \frac{170/3}{100 - 170/3}$$

$$= 1 + 4 + \frac{170}{130} = 5 + \frac{17}{13} = 6.3077 = \frac{82}{13}$$

(b) $$W_s = \frac{L_s}{\lambda} = \frac{82/13}{30} = \frac{82}{13 \times 30} = 0.2103$$

Problems:

① Consider a queueing system where arrivals are according to a Poisson distribution with mean 5/hr. Find the expected waiting time in the system if the service time distribution is (i) Uniform from t=5 min to t=15 min. (ii) Normal with mean 3 mins & variance 4 mins^2.

Sol: This is an (M/G/1) queue model.

Given: Mean arrival rate $\lambda = 5$ per hr $= \frac{5}{60}$ per min $= \frac{1}{12}$ per min.

(i) The service time distribution is uniform from t=5 mins to t=15 mins.

$\therefore a=5,\ b=15$

$$\text{Mean service time} = \frac{a+b}{2} = \frac{5+15}{2} = 10 \text{ mins}$$

$$\text{Mean service rate } \mu = \frac{1}{10} \text{ per min}.$$

$$\sigma^2 = Var(T) = \frac{(b-a)^2}{12} = \frac{(15-5)^2}{12} = 8.33$$

$$\rho = \frac{\lambda}{\mu} = \frac{(1/12)}{(1/10)} = 0.833$$

$$L_s = \left[\frac{\lambda^2\sigma^2+\rho^2}{2(1-\rho)}\right] + \frac{\lambda}{\mu} = \frac{(1/12)^2\, 8.33 + (0.833)^2}{2(1-0.833)} + 0.833 = 3.084$$

$$W_s = \frac{L_s}{\lambda} = \frac{3.084}{1/12} = 12 \times 3.084 = 37 \text{ mins}$$

(ii) The service time distribution is Normal with mean 3 mins & variance 4 mins^2.

Given: Mean arrival rate $\lambda = \frac{1}{12}$ per min.

Mean service time = 3 mins

$$\text{Mean service rate } \mu = \frac{1}{3} \text{ per min.}$$

$$\sigma^2 = 4$$

$$\rho = \frac{\lambda}{\mu} = \frac{(1/12)}{(1/3)} = \frac{1}{4}$$

$$L_s = \frac{\lambda^2\sigma^2+\rho^2}{2(1-\rho)} + \frac{\lambda}{\mu} = \frac{(1/12)^2(4) + (1/4)^2}{2(1-1/4)} + \frac{1}{4}$$

$$= \frac{0.0903}{1.5} + \frac{1}{4} = 0.0602 + \frac{1}{4} = 0.31$$

$W_s = \frac{L_s}{\lambda} = \frac{1}{(1/12)} \times 0.31 = 12 \times 0.31 = 3.72$ mins

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② A car manufacturing plant uses one big crane for loading cars into a truck. Cars arrive for loading by the crane according to a Poisson distribution with a mean of 5 cars per hour. Given that the service time for all cars is constant & equal to 6 mins, determine L_s, L_q, W_s & W_q.

Sol: The given problem is in $(M/G/1):(\infty/FIFO)$ model.

Given: Mean arrival rate $\lambda = 5$ per hr.

Mean service time $= 6$ mins $= \frac{6}{60} = \frac{1}{10}$ hr

Mean service rate $\mu = \frac{1}{1/10}$ per hr $= 10$ per hr.

$$\rho = \frac{\lambda}{\mu} = \frac{5}{10} = 1/2$$

The service time T is constant then $Var(T) = \sigma^2 = 0$

$$L_s = \frac{\lambda^2\sigma^2 + \rho^2}{2(1-\rho)} + \frac{\lambda}{\mu} = \frac{0 + (1/2)^2}{2(1-1/2)} + \frac{1}{2} = \frac{1}{4} + \frac{1}{2} = \frac{3}{4} = 0.75$$

$$L_q = L_s - \frac{\lambda}{\mu} = \frac{3}{4} - \frac{1}{2} = \frac{1}{4} = 0.25$$

$$W_s = \frac{L_s}{\lambda} = \frac{0.75}{5} = 0.15 \text{ hr} = 0.15 \times 60 = 9 \text{ mins}$$

$$W_q = \frac{L_q}{\lambda} = \frac{0.25}{5} = 0.05 \text{ hr} = 0.05 \times 60 = 3 \text{ mins.}$$

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③ A one man barber shop takes exactly 25 mins to complete one haircut. If customer arrive at the barber shop in a Poisson fashion at an average rate of one every 40 mins)., how long on the average a customer spends in the shop? Also find the average time a customer must wait for service.

Sol: Given:

The average arrival time $= 40$ mins

$\lambda = \frac{1}{40}$ per min/.

The average service time $= 25$ mins.

$\mu = \frac{1}{25}$ per min.

$$\rho = \frac{\lambda}{\mu} = \frac{1/40}{1/25} = \frac{5}{8}$$

If the service time T is constant, then $\sigma^2 = 0$.

$$L_s = \frac{\lambda^2\sigma^2 + \rho^2}{2(1-\rho)} + \frac{\lambda}{\mu} = \frac{0 + (5/8)^2}{2(1-5/8)} + \frac{5}{8} = \frac{(25/64)}{2(3/8)} + \frac{5}{8} = \frac{25}{48} + \frac{5}{8} = \frac{55}{48}$$

$$L_q = L_s - \frac{\lambda}{\mu} = \frac{55}{48} - \frac{5}{8} = \frac{25}{48}$$

$$W_s = \frac{L_s}{\lambda} = \frac{55/48}{1/40} = 45.833$$

$$W_q = \frac{L_q}{\lambda} = \frac{25/48}{1/40} = 20.833$$

④ In factory cafteria, the customers have to pass through 3 counters. The customers buy coupons at the 1st counter, select ~~the~~ & collect snacks at the 2nd counter & collect tea at the 3rd counter. The server at each counter takes on an average 1.5 min although the distribution of service time is approximately exponential. If the arrival of customers in approximately Poisson, at an average rate of 6/hr, then calculate:

(i) the average time of the customers spend in the cafteria

(ii) the average time of the customers of getting the service

(iii) the most probable time in getting the service.

Sol: Given $k = 3$

Service time at one counter = 1.5 mins

∴ Total service time = 4.5 mins

$$\mu = \frac{1}{4.5} \times 60 = 13.33 \,/\text{hr}$$

$$\lambda = 6/\text{hr}$$

$$L_s = \frac{\lambda}{\mu} + \frac{\lambda^2}{\mu(\mu-\lambda)}\left(\frac{k+1}{2k}\right) = \frac{6}{13.33} + \frac{6^2}{(13.33)(13.33-6)}\left(\frac{3+1}{2(3)}\right)$$

$$= \frac{6}{13.33} + \frac{36}{97.709}\left(\frac{4}{6}\right) = 0.4501 + 0.2456 = 0.6957$$

$L_q = L_s - \frac{\lambda}{\mu} = 0.6957 - 0.4501 = 0.2456$

$W_s = \frac{L_s}{\lambda} = \frac{0.6957}{6} = 0.11595\ hr$

$W_q = \frac{L_q}{\lambda} = \frac{0.2456}{6} = 0.0409\ hr$

$\text{Mode} = \frac{k-1}{\mu k} = \frac{3-1}{13.33 \times 3} = \frac{2}{39.99} = 0.05\ hr$

(i) Average waiting time customer spend in cafeteria

$W_s = 0.1159\ hr$

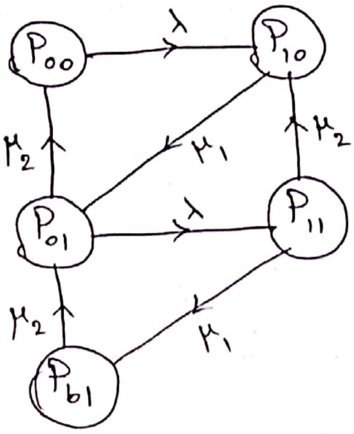
(ii) Average time getting the service

$W = W_s - W_q = 0.115 - 0.0409 = 0.0741\ hr = 0.0741 \times 60 = 4.5$ mins

(iii) Most probable time

Mode $= 0.05\ hr$

State transition diagram:



The balance equn/. in a two state blocked system

State	Outgoing = Incoming
(0,0)	$\lambda P_{00} = \mu_2 P_{01}$
(1,0)	$\mu_1 P_{10} = \lambda P_{00} + \mu_2 P_{11}$
(0,1)	$\lambda P_{01} + \mu_2 P_{01} = \mu_1 P_{10} + \mu_2 P_{b1}$ $\Rightarrow (\lambda + \mu_2) P_{01} = \mu_1 P_{10} + \mu_2 P_{b1}$
(1,1)	$\mu_1 P_{11} + \mu_2 P_{11} = \lambda P_{01} \Rightarrow (\mu_1 + \mu_2) P_{11} = \lambda P_{01}$
(b,1)	$\mu_2 P_{b1} = \mu_1 P_{11}$

& $P_{00} + P_{01} + P_{10} + P_{11} + P_{b1} = 1$

Solving the above eqns/., we get

$$P_{00} = \frac{\mu_1 \mu_2^2 (\mu_1+\mu_2)}{\mu_2(\mu_1+\mu_2)(\mu_1\mu_2+\lambda\mu_1+\lambda\mu_2)+\lambda^2(\mu_1^2+\mu_2^2+\mu_1\mu_2)}$$

$$P_{01} = \frac{\lambda}{\mu_2} P_{00}$$

$$P_{11} = \frac{\lambda^2}{\mu_2(\mu_1+\mu_2)} P_{00}$$

$$P_{10} = \left(\frac{\lambda^2}{\mu_1(\mu_1+\mu_2)} + \frac{\lambda}{\mu_1}\right) P_{00}$$

$$P_{b1} = \frac{\mu_1 \lambda^2}{\mu_2^2(\mu_1+\mu_2)} P_{00}$$

(i) The average no/. of customers in the system

$$L_s = P_{01}+P_{10}+2(P_{11}+P_{b1})$$

(ii) The average waiting time of a customer in the system

$$W_s = \frac{L_s}{\lambda(P_{00}+P_{01})}$$

(iii) The proportion of customers who enter in the shop $= P_{00}+P_{01}$

(iv) Effective arrival rate $= P$(an arriving customer enters the system) $\times \lambda$

$$= (P_{00}+P_{01})\lambda$$

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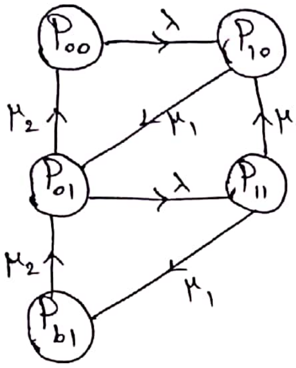
⑤ There are two salesmen in a fruit shop, one incharge of Billing & receiving payment & the other in charge of weighing & delivering the items. Due to limited availability of space, only one customer is allowed to enter the shop, that too when the billing clerk is free. The customer who has finished his billing job has to wait there until the delivery section becomes free. If customer reach the fruit shop according to Poisson process at the rate of 5 per hour & both the salesmen take 6 mins each to serve a customer on the average & the service time follow an exponential distribution, find the average no/. of customers in the shop & the average time spent by a customer who has entered the shop.

Sol: Given $\lambda = 5$ per hr

Mean service time $= 6$ mins $= \frac{6}{60} = \frac{1}{10}$ hr

$\mu_1 = \mu_2 =$ Mean service rate $= \frac{1}{\text{mean service time}} = 1/_{1/10} = 10$ per hr.



The balance equns. in a two state blocked system

State	Outgoing = Incoming
$(0,0)$	$\lambda P_{00} = \mu_2 P_{01}$ —①
$(1,0)$	$\mu_1 P_{10} = \lambda P_{00} + \mu_2 P_{11}$ —②
$(0,1)$	$\lambda P_{01} + \mu_2 P_{01} = \mu_1 P_{10} + \mu_2 P_{b1}$ $\Rightarrow (\lambda+\mu_2) P_{01} = \mu_1 P_{10} + \mu_2 P_{b1}$ —③
$(1,1)$	$\mu_1 P_{11} + \mu_2 P_{11} = \lambda P_{01} \Rightarrow (\mu_1+\mu_2) P_{11} = \lambda P_{01}$ —④
$(b,1)$	$\mu_2 P_{b1} = \mu_1 P_{11}$ —⑤

& $P_{00} + P_{01} + P_{10} + P_{11} + P_{b1} = 1$ —⑥

① $\Rightarrow 5P_{00} = 10P_{01} \Rightarrow P_{00} = 2P_{01}$ —⑦

② $\Rightarrow 10P_{10} = 5P_{00} + 10P_{11} \Rightarrow 2P_{10} = P_{00} + 2P_{11}$ —⑧

③ $\Rightarrow 15P_{01} = 10P_{10} + 10P_{b1} \Rightarrow 3P_{01} = 2P_{10} + 2P_{b1}$ —⑨

④ $\Rightarrow 20P_{11} = 5P_{01} \Rightarrow 4P_{11} = P_{01}$ —⑩

⑤ $\Rightarrow 10P_{b1} = 10P_{11} \Rightarrow P_{b1} = P_{11}$ —⑪

To find: P_{00}, P_{10}, P_{01}, P_{11} & P_{b1}

⑦ $\Rightarrow P_{01} = \frac{1}{2} P_{00}$ —⑫

⑩ $\Rightarrow P_{11} = \frac{1}{4} P_{01} = \frac{1}{4}\left(\frac{1}{2} P_{00}\right) = \frac{1}{8} P_{00}$ —⑬

⑪ $\Rightarrow P_{b1} = P_{11} = \frac{1}{8} P_{00}$ —⑭

⑧ $\Rightarrow P_{10} = \frac{1}{2}\left[P_{00} + 2P_{11}\right] = \frac{1}{2}\left[P_{00} + 2\left(\frac{1}{8}\right)P_{00}\right] = \frac{5}{8} P_{00}$ —⑮

⑥ $\Rightarrow P_{00} + P_{10} + P_{01} + P_{11} + P_{b1} = 1$

$\Rightarrow P_{00} + \frac{5}{8} P_{00} + \frac{1}{2} P_{00} + \frac{1}{8} P_{00} + \frac{1}{8} P_{00} = 1$

$P_{00}\left(\frac{19}{8}\right) = 1 \Rightarrow P_{00} = \frac{8}{19}$

⑫ $\Rightarrow P_{01} = \frac{1}{2}\left(\frac{8}{19}\right) = \frac{4}{19}$

⑭ $\Rightarrow P_{b1} = \frac{1}{8}\left(\frac{8}{19}\right) = \frac{1}{19}$

⑬ $\Rightarrow P_{11} = \frac{1}{8}\left(\frac{8}{19}\right) = \frac{1}{19}$

⑮ $\Rightarrow P_{10} = \frac{5}{8}\left(\frac{8}{19}\right) = \frac{5}{19}$

$$L_s = P_{01} + P_{10} + 2(P_{11} + P_{b1}) = \frac{4}{19} + \frac{5}{19} + 2\left(\frac{1}{19} + \frac{1}{19}\right) = \frac{13}{19}$$

$$W_s = \frac{L_s}{\lambda(P_{00}+P_{01})} = \frac{(13/19)}{5\left(\frac{8}{19}+\frac{4}{19}\right)} = \frac{13}{60} hr$$

Note: To find the blocking time

Actual service time $= 6+6 = 12$ mins

Blocking time $= W_s - 12 = 13 - 12 = 1$ min.

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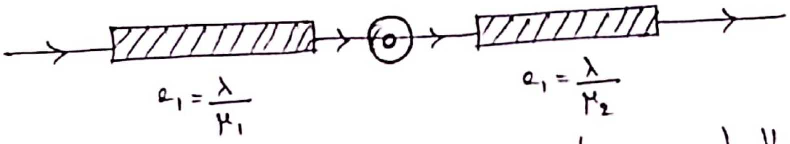
⑥ Consider a 2 stage tandem queue with external arrival rate λ to node 0. Let μ_0 & μ_1 be the service rates of the exponential servers at node 0 & 1 respectively. Arrival process is Poisson. Model this system using a Markov chain & obtain the balance eqns..

Sol:



State	Rate that the process leaves = rate that it enters
$(0,0)$	$\lambda P_{00} = \mu_2 P_{01}$
$(n,0),(n>0)$	$(\lambda+\mu_1)P_{n0} = \mu_2 P_{n1} + \lambda P_{n-1,0}$
$(0,m),(m>0)$	$(\lambda+\mu_2)P_{0m} = \mu_2 P_{0,m+1} + \mu_1 P_{1,m-1}$
$(n,m)\ (nm>0)$	$(\lambda+\mu_1+\mu_2)P_{nm} = \mu_2 P_{n,m+1} + \mu_1 P_{n+1,m-1} + \lambda P_{n-1,m}$

Also $\sum_{n,m} P_{n,m} = 1$

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⑦ There are 2 salesmen in a ration shop one incharge of billing & receiving payment & the other incharge of weighing & delivering the items. Due to limited availability of space, only one customer is allowed to enter the shop, that too when the billing clerk is free. The customer who has finished his billing job has to wait there until the delivery section becomes free. If customers arrive in accordance with a Poisson process at rate 1 & the service times of 2 clerks are independent & have exponential rate of 3 & 2 find

(i) the proportion of customers who enter the ration shop.

(ii) the average no. of customers in the shop.

(iii) The average amount of time that an entering customer spends in the shop..

Sol:

Given $\lambda=1,\ \mu_1=3,\ \mu_2=2.$

The balance equn. in a two state blocked system:

State	Outgoing = Incoming
$(0,0)$	$\lambda P_{00}=\mu_2 P_{01}$ — ①
$(1,0)$	$\mu_1 P_{10}=\lambda P_{00}+\mu_2 P_{11}$ — ②
$(0,1)$	$\lambda P_{01}+\mu_2 P_{01}=\mu_1 P_{10}+\mu_2 P_{b1} \Rightarrow (\lambda+\mu_2)P_{01}=\mu_1 P_{10}+\mu_2 P_{b1}$ — ③
$(1,1)$	$\mu_1 P_{11}+\mu_2 P_{11}=\lambda P_{01} \Rightarrow (\mu_1+\mu_2)P_{11}=\lambda P_{01}$ — ④
$(b,1)$	$\mu_2 P_{b1}=\mu_1 P_{11}$ — ⑤

& $P_{00}+P_{01}+P_{10}+P_{11}+P_{b1}=1$ — ⑥

① $\Rightarrow P_{00}=2P_{01}$ — ⑦

② $\Rightarrow 3P_{10}=P_{00}+2P_{11}$ — ⑧

③ $\Rightarrow 3P_{01}=3P_{10}+2P_{b1}$ — ⑨

④ $\Rightarrow 5P_{11}=P_{01}$ — ⑩

⑤ $\Rightarrow 2P_{b1}=3P_{11}$ — ⑪

⑦ $\Rightarrow P_{01}=\frac{1}{2}P_{00}$ — ⑫

⑩ $\Rightarrow P_{11}=\frac{1}{5}P_{01}=\frac{1}{5}\left(\frac{1}{2}P_{00}\right)=\frac{1}{10}P_{00}$ — ⑬

⑪ $\Rightarrow P_{b1}=\frac{3}{2}P_{11}=\frac{3}{2}\left(\frac{1}{10}P_{00}\right)=\frac{3}{20}P_{00}$ — ⑭

⑧ $\Rightarrow P_{10}=\frac{1}{3}P_{00}+\frac{2}{3}P_{11}=\frac{1}{3}P_{00}+\frac{2}{3}\left(\frac{1}{10}P_{00}\right)=\left(\frac{1}{3}+\frac{1}{15}\right)P_{00}=\frac{2}{5}P_{00}$ — ⑮

⑥ $\Rightarrow P_{00}+P_{01}+P_{10}+P_{11}+P_{b1}=1$

$$P_{00}+\frac{1}{2}P_{00}+\frac{2}{5}P_{00}+\frac{1}{10}P_{00}+\frac{3}{20}P_{00}=1$$

$$\left(\frac{43}{20}\right)P_{00}=1 \Rightarrow P_{00}=\frac{20}{43}$$

$$\therefore P_{01}=\frac{10}{43},\quad P_{11}=\frac{2}{43},\quad P_{b1}=\frac{3}{43},\quad P_{10}=\frac{8}{43}$$

(i) The proportion of customers who enter the ratio shop $=P_{00}+P_{01}=\frac{20}{43}+\frac{10}{43}=\frac{30}{43}$

(ii) $L_s=P_{01}+P_{10}+2(P_{11}+P_{b1})=\frac{10}{43}+\frac{8}{43}+2\left(\frac{2}{43}+\frac{3}{43}\right)=\frac{28}{43}$

(iii) $W_s=\dfrac{L_s}{\lambda(P_{00}+P_{01})}=\dfrac{28/43}{30/43}=\frac{14}{15}$ hrs $=56$ mins

⑧ In a big factory, there are a large no. of operating machines & two sequential repair shops, which do the service of the damaged machines exponentially with respective rates of 1/hr & 2/hr. If the cumulative failure rate of all the machines in the factory is 0.5/hr, find (i) the probability that both repairshops are idle (ii) the average no. of machines in the service section of the factory & (iii) the average repair time of a machine.

Sol:

Given $\lambda = 0.5/hr = \frac{1}{2}$ per hr.

$\mu_1 = 1$ per hr. , $\mu_2 = 2$ per hr.

The situation in this problem is comparable with 2-stage Tandem queue with single server at each state.

(i) P(both the service stations are idle) $= P(0,0)$

$$= \left(\frac{\lambda}{\mu_1}\right)^0 \left(1-\frac{\lambda}{\mu_1}\right) \left(\frac{\lambda}{\mu_2}\right)^0 \left(1-\frac{\lambda}{\mu_2}\right)$$

$$= \left(1-\frac{1/2}{1}\right)\left(1-\frac{1/2}{2}\right) = \frac{1}{2} \times \frac{3}{4} = \frac{3}{8}$$

(ii) The average no. of machines in service $= \frac{\lambda}{\mu_1-\lambda} + \frac{\lambda}{\mu_2-\lambda}$

$$= \frac{1/2}{1-1/2} + \frac{1/2}{2-1/2} = 1 + 1/3 = 4/3$$

(iii) The average repair time $= \frac{1}{\mu_1-\lambda} + \frac{1}{\mu_2-\lambda} = \frac{1}{1-1/2} + \frac{1}{2-1/2}$

$$= 2 + \frac{2}{3} = \frac{8}{3}$$